Closed expression of the hyper-complex Fourier kernel

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When eating bamboo sprouts, remember the man who planted them.

Chinese Proverb

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The deep study of nature is the most fruitful source of mathematical discoveries.

Baron J. Fourier

1 Introduction

The classical Fourier transform has been among the most widely used tools in sciences and many engineering disciplines. However, as the type of data has evolved tremendously over the past years, the classical Fourier transform meets some new challenges. One recurrent problem is how to represent and analyze multi-channel signals. To solve this problem, several hyper-complex Fourier transforms have been introduced to treat multi-channel signals as a algebraic whole, see [6–9], [57–60], [85–88], [34, 37, 45, 47, 54]. One particular advantage of these transforms is mixing the signals properly because of the underlying algebra structure. For the historical development of hyper-complex Fourier transforms, we refer to [16] and the more recent review [23]. Another challenge problem is how to analyze the data collected on a surface or even more generally a manifold. The problem in these cases is the geometry. One class of hyper-complex Fourier transforms which has attracted quite a lot interest is the so-called Clifford-Fourier transform and its generalizations. These integral transforms are closely related with the operator realization of the Lie algebra $\mathfrak{sl}_2$. The main aim of this thesis is to study and to develop new methods to compute the closed kernel of the Clifford-Fourier transform and other Fourier transforms related with $\mathfrak{sl}_2$, such as the Dunkl transform, as well as develop analogues on non-Euclidean space.

The Clifford-Fourier transform works within the context of Clifford analysis. Generally speaking, Clifford analysis is the function theory of the Dirac operator where the functions take value in a real or complex Clifford algebra or spinor space. It is considered as a direct higher-dimensional generalization of the theory of holomorphic functions in the complex plane and as an elegant refinement of classical harmonic analysis. We refer to [12, 32] for more details of this function theory.

The Clifford-Fourier transform is designed to be a non-scalar integral transform which mixes multi-channel signals. It nevertheless still satisfies many important properties of the classical Fourier transform, such as the Helmholtz relations, inversion, Plancherel theorem and uncertainty principle, etc. One elegant way to characterize this transform is by the operator exponential from which the spectrum and the eigenfunctions could be computed easily. This methods stems from the representation theory of SL_2 where the classical Fourier transform realizes a unitary representation. Also the operator realization of the Lie algebra $\mathfrak{sl}_2$ connects the Hamiltonion operator of the classical harmonic oscillator, where the operator realization is given by the Laplacian Δ, $|x|^2$ and the Euler operator, see [49, 61]. It is worth pointing out that the operator exponential also leads to the fractional Fourier transform [76]. The problem left is to determine the kernel function for this integral transform. Note that in the scalar case, the closed form of the fractional Fourier kernel which is a series of products of Hermite functions is given by the Mehler formula [11]. However in the Clifford case, the explicit computation of the generalized Fourier kernel is not a easy problem.

Let us give a brief review of the computation on the Clifford-Fourier kernel here. The Clifford-Fourier transform was first defined in [13] by acting with $e^{\mp \frac{i\pi}{2}\Gamma_x}$ on the classical Fourier transform. This can be expressed using exponential operators as

$$\mathcal{F}_{\pm} = e^{\frac{i\pi m}{4}} e^{\mp \frac{i\pi}{2}\Gamma_x} e^{\frac{i\pi}{4}(\Delta - |x|^2)}.$$

Then, the Clifford-Fourier kernel of dimension 2 was first constructed by Clifford analysis techniques in [14]. Later, for higher even dimensions, by a complicated iterative procedure, the kernel can be constructed but could only be used practically in low dimensions, see [15]. The explicit formulas in all even dimensions was obtained by De Bie and Xu in [29]. They showed that the even dimensional kernels can be expressed by Bessel function. In later work, the results were extended to fractional versions of the Clifford-Fourier transform [26,73] and integral kernels satisfying certain generalized Helmholtz PDEs in Clifford analysis [25]. A new method based on solving the wave equation on the sphere was given in [73]. The information of the odd dimensional kernels is still very limited. In [28] an approach using Lie superalgebras and group symmetries led to a complete classification of transforms that behave in the same way as both the Clifford-Fourier transform and the classical Fourier transform. Again, only the explicit computation of the even dimensional kernels is known. The main aim of this thesis is to develop new method to compute these generalized kernels.

It is obvious that the Fourier transform and the structure of the space where the transform lives have close relations. On the one hand, the Fourier transform can be used to explore the symmetry of the space [92]. On the other hand, the geometry of the space determines the Fourier transform. The hyperbolic space which has constant sectional curvature -1, of course has quite different geometry than the usual Euclidean space. The Fourier transform on the hyperbolic space was defined by S. Helgason and decomposes any function in the space $L^2(H^m)$ into eigenfunction of the Laplace operator on the hyperbolic space, see [55]. The natural question is how to generalize the Clifford-Fourier transform to the hyperbolic space. It will further help to study the structure of the hyperbolic space. But the missing of the operator realization of the $\mathfrak{sl}_2$ makes it mysterious. Another aim of this thesis is therefore to define the hyper-complex Fourier transform on the hyperbolic space and further determine the generalized Fourier kernel.

Besides the classical operator realization of the $\mathfrak{sl}_2$, there exist others on $\mathbb{R}^m$. Each of them has led to a new integral transform on $\mathbb{R}^m$, such as the Dunkl transform, the (κ, a)-generalized Fourier transform, and further generalized in the Clifford setting, see [27]. However, at the beginning, the Dunkl transform was not defined by the operator exponential but by the differential relations satisfied by the Dunkl operator T_j,

$$T_j E_\kappa(x,y) = -iy_j E_\kappa(x,y), \qquad j = 1, \dots, m.$$

These operators T_j are commuting differential-difference operators associated to a finite reflection group on a Euclidean space. They were introduced by C.F.Dunkl in the late 80ies in [39]. Now, it has become the key tool in the study of special functions with reflection symmetries, see [39–43] [78–84]. It also has deep relations with the analysis of quantum many body systems of Calogero-Moser-Sutherland type [80]. The joint eigenfunction $E_\kappa(x, y)$ is called the Dunkl kernel [30, 31], which is the exponential function $e^{-i\langle x,y\rangle}$ when $\kappa = 0$. The explicit expression of the Dunkl kernel and other generalizations based on the Dunkl operator is only known in a few cases. We will show that our Laplace transform method will extend these results to new cases.

In the remainder of this introductory chapter, we will give an overview of the contents of this thesis.

We start with the preliminaries in Chapter 2. We introduce the Clifford algebra and Euclidean Clifford analysis which form the framework of the Clifford-Fourier transform. As our method works in the Laplace domain, we also introduce the Laplace transform and list the transform formulas needed in this thesis. The hyper-complex Fourier transforms studied in this thesis are defined using the operator exponential which is a generalization of the classical Fourier transform. We introduce the operator exponential formulation of the classical Fourier transform at the end of this chapter.

In Chapter 3, the Laplace transform method is introduced to compute the Clifford-Fourier kernel. This will be done by introducing an auxiliary variable t and subsequently expressing the classical Fourier kernel by the Cauchy kernel and the Szegö kernel in the Laplace domain. Then the action of the exponential operator is understood using the monogenic expansion of the two kernels. The Laplace inversion yields the closed expression of the even dimensional kernels. Moreover, we are able to compute the generating function of the even dimensional kernels. The new method also recovers the plane wave expansion of the kernels given using the previous method in [29]. For the odd dimension, a new integral formula is obtained. Unfortunately, the bounds of the odd dimensional kernel are still open.

The Clifford-Fourier transform is further generalized in [28] using the representation theory of the $\mathfrak{sl}_2$. In Chapter 4, we focus on further developing the Laplace transform method to compute the generalized Fourier kernel. In this chapter, the connection between the kernel of the Clifford-Fourier transform and the generalized Clifford-Fourier transform will be established. This allows us to find the explicit expression of the kernel and the generating function of the even dimensional kernels. Furthermore, we will determine which polynomials G give rise to polynomially bounded kernels. This offers new perspectives to define odd dimensional hyper-complex Fourier transforms.

The representation and analysis of signals in non-Euclidean geometry is now a recurrent problem in many scientific domains. A lot of efforts have been devoted to this problem, see [1,2,10]. Chapter 5 will concentrate on developing the Clifford-Fourier transform on the hyperbolic space. The new transform will be defined by acting with the exponential of the spherical Dirac operator on the Helgason Fourier kernel which is based on the geometry property, i.e. the geodesic sphere in H^m is a Euclidean sphere. We further find the correspondence between the Laplace domain results of the Euclidean case and the hyperbolic case. This chapter establishes these similarities and proves new results on this generalized Fourier transform.

Chapter 6 is devoted to develop the Laplace method to obtain explicit and integral expressions for the kernel of the (κ, a)-generalized Fourier transform for $\kappa = 0$. In the case of dihedral groups, this method will also be applied to the Dunkl kernel as well as the Dunkl Bessel function. By making use of the Poisson kernel, the kernel in the Laplace domain takes on a much simpler form. The inverse Laplace transform can then be computed using the generalized Mittag-Leffler function to obtain integral expressions. In case the parameters involved are integers, explicit formulas will be obtained using partial fraction decomposition. New bounds for the kernel of the (κ, a)-generalized Fourier transform are obtained as well.

We end this work with our conclusions and some open problems.

Residues arise... naturally in several branches of analysis.... Their consideration provides simple and easy-to-use methods, which are applicable to a large number of diverse questions, and some new results...

Augustin Louis Cauchy

2 Preliminaries

This chapter is organized as follows. In the first section, we introduce the Clifford algebra over $\mathbb{R}^m$ together with some properties. Euclidean Clifford analysis then follows in Section 2.2. Here the Dirac operator, monogenicity and the spherical Dirac operator are introduced. Furthermore, the Taylor series expansion of a monogenic function is introduced which plays an important role in our Laplace method. In Section 2.3, we give a brief introduction of the Laplace transform and list the transform formulas which will be used in this thesis. The standard reference of the Laplace transform is [36]. The Clifford-Fourier transform and its generalization are defined by operator exponentials which are the generalization of the classical results. In Section 2.4, we give a short review of the operator exponential characterization of the classical Fourier transform.

2.1 Clifford algebra

In this section, we give a brief introduction of the Clifford algebra over $\mathbb{R}^m$. For the in-depth study of the existence and construction of the universal Clifford algebra, we refer to [12] and [51].

Let $\mathbb{R}^m$ be the usual m-dimensional Euclidean space with an orthonormal basis $\{e_1, e_2, \ldots, e_m\}$. The real Clifford algebra $\mathcal{C}\ell_{0,m}$ associated with $\mathbb{R}^m$ is spanned by the reduced products

$$\bigcup_{j=1}^{m}\{e_\alpha = e_{i_1}e_{i_2}\ldots e_{i_j} : \alpha = \{i_1, i_2, \ldots, i_j\}, \quad 1 \le i_1 < i_2 < \cdots < i_j \le m\}$$

with the relations $e_ie_j + e_je_i = -2\delta_{ij}$. We have

$$\mathcal{C}\ell_{0,m} = \left\{\sum_\alpha e_\alpha x_\alpha; x_\alpha \in \mathbb{R}\right\}.$$

The Clifford algebra $\mathcal{C}\ell_{0,m}$ is a graded algebra as

$$\mathcal{C}\ell_{0,m} = \oplus_{l=0}^{m}\mathcal{C}\ell_{0,m}^{(l)}$$

where $\mathcal{C}\ell_{0,m}^{(l)}$ is spanned by reduced Clifford products of length l. By the canonical mapping $x = \sum_{j=1}^{n} x_je_i$, the 1-vector space $\mathcal{C}\ell_{0,m}^{(1)}$ is isomorphic with $\mathbb{R}^m$. Furthermore, we point out that the Clifford algebra $\mathcal{C}\ell_{0,m}$ is a 2^m-dimensional real associative algebra with identity.

For any $x, y \in \mathcal{C}\ell_{0,m}$, the conjugation is defined by

$$\overline{(e_{j_1}\ldots e_{j_l})} = (-1)^l e_{j_l}\ldots e_{j_1}$$

as a linear mapping. We have $\overline{(xy)} = \overline{y}\,\overline{x}, \overline{\overline{x}} = x$.

Now we can define the inner product and the wedge product of two vectors $x, y \in \mathbb{R}^m$ using the Clifford product:

$$\begin{aligned}\langle x, y\rangle : &= \sum_{j=1}^{m} x_jy_j = -\frac{1}{2}(xy + yx);\\ x \wedge y : &= \sum_{j<k} e_je_k(x_jy_k - x_ky_j) = \frac{1}{2}(xy - yx).\end{aligned}$$

By the definitions, it is easy to get $xy = -\langle x, y\rangle + x \wedge y$, $(x \wedge y)^2 = -|x|^2|y|^2 + \langle x, y\rangle^2$ and $|x|^2 = x\overline{x} = -x^2$ (see also [29]). We consider $\frac{x\wedge y}{|x\wedge y|}$ as an imaginary unit because $\frac{(x\wedge y)^2}{|x\wedge y|^2} = -1$.

The complexified Clifford algebra $\mathcal{C}\ell_{0,m}^{c}$ is defined as $\mathbb{C} \otimes \mathcal{C}\ell_{0,m}$.

2.2 Clifford analysis

Functions defined on $\mathbb{R}^m$, taking values in the Clifford algebra, can be written as $f = \sum_\alpha e_\alpha f_\alpha(x_1, \cdots, x_m)$. The most important operator in Clifford analysis is the Dirac operator which is a vector valued first order differential operator

$$D = \sum_{j=1}^{m} e_j \partial_{x_j}.$$

The Dirac operator is a rotation-invariant operator which means that it is invariant under the special orthogonal group $SO(m)$ which is doubly covered by the spin group $Spin(m)$ of the Clifford algebra $\mathcal{C}\ell_{0,m}$. When u is a scalar C^1 function, Du can be identified with the gradient ∇u. A function is called monogenic if $Du = 0$. It is worth pointing out that the Dirac operator factorizes the Laplace operator in the sense that

$$\Delta = -D^2$$

whence the monogenicity may also be regarded as a refinement of harmonicity.

The Dirac operator together with the vector variable x and the Euler operator, satisfy the relations

$$D^2 = -\Delta, \qquad x^2 = -|x|^2, \qquad \{x, D\} = -2\mathbb{E} - m,$$

where $\{a, b\} = ab + ba$ and $\mathbb{E} = \sum_{j=1}^{m} x_j \partial_{x_j}$ is the Euler operator and hence they generate a realization of the Lie superalgebra $\mathfrak{osp}(1|2)$, which contains the Lie algebra $\mathfrak{sl}_2 = \text{span}\{\Delta, |x|^2, [\Delta, |x|^2]\}$ as its even part. This algebraic structure plays a crucial role in defining hyper-complex Fourier transforms.

An important example of a monogenic function is the generalized Cauchy kernel [12]

$$G(x) = \frac{1}{\omega_m} \frac{\bar{x}}{|x|^m}$$

where ω_m is the surface area of the unit ball in $\mathbb{R}^m$. It is the fundamental solution of the Dirac operator.

Denote by $\mathcal{P}$ the space of polynomials taking values in $\mathcal{C}\ell_{0,m}$, i.e. $\mathcal{P} := \mathbb{R}[x_1, \ldots, x_m] \otimes \mathcal{C}\ell_{0,m}$. The space of homogeneous polynomials of degree k is then denoted by $\mathcal{P}_k$. The space $\mathcal{M}_k := (\ker D) \cap \mathcal{P}_k$, is called the space of spherical monogenics of degree k.

The local behaviour of a monogenic function near a point can be investigated by the monogenic polynomials introduced above. The following theorem is the analogue of the Taylor series in complex analysis.

Theorem 2.1. *[12] Suppose f is monogenic in an open set Ω containing the origin. Then there exists an open neighbourhood Λ of the origin in which f can be developed into a normally convergent series of spherical monogenics $M_k f(x)$, i.e.*

$$f(x) = \sum_{k=0}^{\infty} M_k f(x),$$

with $M_k f(x) \in \mathcal{M}_k$.

The other main operator in this thesis is the Gamma operator or the angular Dirac operator (see [12])

$$\Gamma_x := -\sum_{j<k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j}) = -x D_x - E_x.$$

Here $E_x = \sum_{i=1}^{m} x_i \partial_{x_i}$ is the Euler operator. One property of the Γ_x which will be used frequently is that it commutes with scalar radial functions. The operator Γ_x has two important eigenspaces:

$$\Gamma_x \mathcal{M}_k = -k \mathcal{M}_k, \tag{2.1}$$

$$\Gamma_x (x \mathcal{M}_{k-1}) = (k+m-2) x \mathcal{M}_{k-1} \tag{2.2}$$

which follows from the definition of Γ_x.

In the following, we introduce the Casimir operator of $Spin(m)$ representation. A basic spinor representation $\mathbb{S}$ is a complex irreducible representation of the Clifford algebra $\mathcal{C}\ell_{0,m}$. If m is even, as a $Spin(m)$-module, $\mathbb{S}$ decomposes into two irreducible inequivalent pieces. When m is even, the spinor space $\mathbb{S}$ can be realised as a minimal left ideal of $\mathcal{C}\ell_{0,m}^{c}$. In this thesis, we will use $\mathcal{C}\ell_{0,m}^{c}$ as a representation space of $Spin(m)$. Let h and l stand for the vectorial representation of $Spin(m)$ on S^{m-1} and for the representation of $Spin(m)$ on $\mathcal{C}\ell_{0,m}^{c}$ obtained by left multiplication. Then these representations give rise to representations H and L of $Spin(m)$ on $L^2(S^{m-1}, \mathcal{C}\ell_{0,m}^{c})$. The Casimir operator of the corresponding infinitesimal representation is

$$C(H) = \Delta_{S^{m-1}}, \qquad C(L) = \Delta_{S^{m-1}} + \Gamma - \frac{1}{4}\binom{m}{2}.$$

Both Casimir operators are polynomials in the operator Γ [32].

For more function theory of the Dirac operator, we refer to [12].

2.3 The Laplace transform

The main tool of this thesis is the Laplace transform . In this section, we give a brief introduction of this transform and list some transform tables we will use in the following chapters. For more details of the Laplace transform, we refer to [36].

The Laplace transform is an integral transform which takes a function of a positive real variable t to a function of a complex variable s. For a function $f(t)$ which has exponential order $|f(t)| \leq Ce^{\alpha t}, t \geq t_0$, the Laplace transform is defined as

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

By Lerch's theorem [89], if we restrict our attention to functions which are continuous on $[0, \infty)$, then the inverse transform

$$\mathcal{L}^{-1}(F(s)) = f(t)$$

is uniquely defined. The inverse Laplace transform is given by the Bromwich integral or Post's inversion formula. In practice, it is typically more convenient to decompose a Laplace transform into known transforms of functions obtained from a table, for example [48].

We give the following Laplace transform formulas which will be used in this thesis with $r = (s^2+a^2)^{1/2}$, $R = s+r$ and $g(s) = \mathcal{L}(f(t))$ see ([48]):

$$\begin{aligned}
\mathcal{L}(t^{k-1}) &= \frac{\Gamma(k)}{s^k}, \quad k > 0, \quad \mathrm{Re}(s) > 0; \\
\mathcal{L}(e^{-\alpha t}) &= \frac{1}{s+\alpha}, \quad \mathrm{Re}(s) > -\mathrm{Re}(\alpha); && (2.3) \\
\mathcal{L}(t^{k-1}e^{-\alpha t}) &= \frac{\Gamma(k)}{(s+\alpha)^k}, \quad k > 0, \quad \mathrm{Re}(s) > -\mathrm{Re}(\alpha); && (2.4) \\
\mathcal{L}(\cos at) &= \frac{s}{s^2+a^2}, \quad \mathrm{Re}(s) > |\mathrm{Im}(a)|; && (2.5) \\
\mathcal{L}(\sin at) &= \frac{a}{s^2+a^2}, \quad \mathrm{Re}(s) > |\mathrm{Im}(a)|; && (2.6) \\
\mathcal{L}(J_\nu(at)) &= \frac{1}{r}\left(\frac{a}{R}\right)^\nu, \quad \mathrm{Re}(\nu) > -1, \mathrm{Re}(s) > |\mathrm{Im}(a)|; && (2.7)
\end{aligned}$$

and

$$\mathcal{L}(t^\nu J_\nu(at)) = 2^\nu \Gamma(\nu + \frac{1}{2}) \pi^{-1/2} a^\nu r^{-2\nu-1},$$

$$\operatorname{Re}(\nu) > -1/2, \operatorname{Re}(s) > |\operatorname{Im}(a)|; \quad (2.8)$$

$$\mathcal{L}\Big(t^{\nu+1}J_\nu(at)\Big) = \frac{2^{\nu+1}\Gamma(\nu+3/2)}{\pi^{1/2}} a^\nu r^{-2\nu-3} s,$$
$$\operatorname{Re}(\nu) > -1, \operatorname{Re}(s) > |\operatorname{Im}(a)|; \quad (2.9)$$

We also need some inverse Laplace transforms formulas,

$$\mathcal{L}^{-1}(r^{-1}g(r)) = \int_0^t J_0[a(t^2-u^2)^{1/2}]f(u)du; \quad (2.10)$$

$$\mathcal{L}^{-1}(g(r)) = f(t) - a\int_0^t f[(t^2-u^2)^{1/2}]J_1(au)du; \quad (2.11)$$

$$\mathcal{L}^{-1}(sr^{-1}g(r)) = f(t) - at\int_0^t (t^2-u^2)^{-1/2} \times J_1[a(t^2-u^2)^{1/2}]f(u)du. \quad (2.12)$$

Let $G(s) = \mathcal{L}(g(t))$ and $F(s) = \mathcal{L}(f(t))$. We have the convolution formula for the Laplace transform

$$G(s)F(s) = \mathcal{L}\Big(\int_0^t g(t-\tau)f(\tau)d\tau\Big) \quad (2.13)$$

Another technique frequently used for the inversion of the Laplace transform is the partial fraction decomposition. The partial fraction decomposition of a rational polynomial

$$F(s) = \frac{\sum_{k=0}^m a_k s^k}{\sum_{j=0}^n b_j s^j} = \frac{A(s)}{B(s)}, \qquad (n > m)$$

expresses $F(s)$ as a sum of fractions with simple denominator. We only show the case when $F(s)$ has a single pole of order m. Then $F(s)$ can be expressed as

$$F(s) = \frac{A(s)}{(s-p)^m} = \sum_{k=1}^m \frac{c_k}{(s-p)^k},$$

with complex constants

$$c_{m-k} = \frac{1}{k!}\frac{d^{m-k}}{ds^{m-k}}[F(s)(s-p)^m]_{s=p}, k = 1, \cdots, m.$$

The Laplace transform of a matrix valued function is simply the matrix of Laplace transforms of the individual elements. For example

$$\mathcal{L}\begin{pmatrix} e^t \\ te^{-t} \end{pmatrix} = \begin{pmatrix} 1/(s-1) \\ 1/(s+1)^2 \end{pmatrix}.$$

Suppose A is an $n \times n$ matrix. The matrix exponential is interpreted in terms of a power series, namely

$$\exp(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

By analogy with the scalar case, we have

$$\mathcal{L}(e^{At}) = (sI - A)^{-1}.$$

For more about the Laplace transform of matrix-valued functions, see [96].

2.4 Classical Fourier transform

For $x, y \in \mathbb{R}^m$, the classical Fourier transform in $\mathbb{R}^m$ is defined by

$$\mathcal{F}[f(x)](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle} f(x) dx$$

with $\langle x, y\rangle = \sum_{j=1}^{m} x_j y_j$ the Euclidean inner product. This transform is an isomorphism on the space $\mathcal{S}(\mathbb{R}^m)$. The inverse transform is given by

$$\mathcal{F}^{-1}[f(x)](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{i\langle x,y\rangle} f(x) dx.$$

Let us state the eigenfunctions of this transform. For that aim, we introduce the Hermite functions now,

$$\phi_{j,k,l} := 2^j j! L_j^{\frac{m}{2}+k-1}(|x|^2) H_k^{(l)} e^{-|x|^2/2},$$

where $j, k \in \mathbb{Z}_{\geq 0}$, L_j^{α} is a generalized Laguerre polynomial and $\{H_k^{(l)} | l = 1, \cdots, dim(\mathcal{H}_k)\}$ is a basis for $\mathcal{H}_k$, the space of spherical harmonics of degree k [93]. The Hermite functions form the eigenfunctions of the Fourier transform with the eigenvalue

$$\mathcal{F}(\phi_{j,k,l}) = (-i)^{2j+k} \phi_{j,k,l}.$$

Alternatively, the Hermite functions are also the eigenfunctions of the Hamilton of the harmonic oscillator $\mathcal{H} := \frac{1}{2}(-\Delta + |x|^2 - m)$, with eigenvalues given by

$$\mathcal{H}[\phi_{j,k,l}](y) = (2j + k)\phi_{j,k,l}.$$

Now, in the sense of having the same eigenfunctions and eigenvalues, an equivalent formulation for the Fourier transform is given by the operator exponential

$$\mathcal{F} = e^{-i\frac{\pi}{2}\mathcal{H}} = e^{i\frac{\pi}{4}(\Delta - |x|^2 + m)}.$$

This formulation stems from the representation theory of the Lie algebra $\mathfrak{sl}_2$ generated by Δ, $|x|^2$ and connects with the theory of the quantum harmonic oscillator, see [61] [49]. It allows to easily compute the spectrum of the Fourier transform. This operator plays a key role in this thesis. New generalized transforms are defined using the operator exponential because many important properties can be obtained immediately from the operator exponential such as inversion, Plancherel theorem, behaviour of differentiation, etc.

No simplicity of mind, no obscurity of station, can escape the universal duty of questioning all that we believe.

William K. Clifford

3
Clifford-Fourier kernel

The Clifford-Fourier transform was introduced in [13] as a generalization of the classical Fourier transform (FT) for multichannel signals. The main aim of the present chapter is to develop a new and elegant method to compute its integral kernel K_m. This will be done by introducing an auxiliary variable t and subsequently expressing its Laplace transform

$$\mathcal{L}(t^{m/2-1}e^{-it\langle x,y\rangle})$$

in terms of the Cauchy kernel for the Dirac operator. In the Laplace domain, the action of Γ_y is obtained using a monogenic expansion. Laplace inversion then yields our main result in the even dimensional case. As an additional bonus, we are now able to compute an explicit generating function for all even dimensional kernels by again using the Laplace domain expression. This is achieved in Theorem 6.9 and 3.7.

This chapter is organized as follows. In Section 3.1 we recall basic facts concerning the Clifford-Fourier transform. In Section 3.2 we first compute the Laplace domain expression for the fractional Clifford-Fourier kernel. We use this result to reobtain both the plane wave decomposition and the explicit expression through Laplace inversion. Finally we derive a new integral identity for the kernel in odd dimensions and we construct the generating function.

3.1 The Clifford-Fourier transform

In Section 2.4, we explained that the classical Fourier transform

$$\mathcal{F}(f)(y) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle} f(x)dx,$$

with $\langle x, y\rangle$ the inner product on $\mathbb{R}^m$, can be represented by the operator exponential

$$\mathcal{F} = e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta - |x|^2)}.$$

The Clifford Fourier transform was introduced by Brackx, De Schepper and Sommen using the angular Dirac operator Γ_x in the Clifford algebra setting in [13]. More precisely, it is defined by

$$\mathcal{F}_{\pm} = e^{\frac{i\pi m}{4}} e^{\mp \frac{i\pi}{2}\Gamma_x} e^{\frac{i\pi}{4}(\Delta - |x|^2)}.$$

For F_{-}, we denote the kernel as

$$K_m(x, y) = e^{i\frac{\pi}{2}\Gamma_y} e^{-i\langle x,y\rangle}.$$

In general, it is not easy to compute this kernel explicitly. In [29], the authors derived the kernel for even dimensions as a finite sum of Bessel functions. Later in [26], the fractional Clifford-Fourier transform was introduced as a generalization of the fractional Fourier transform

$$\mathcal{F}_{\alpha,\beta} = e^{\frac{i\alpha m}{2}} e^{i\beta\Gamma_x} e^{\frac{i\alpha}{2}(\Delta - |x|^2)}$$

and the kernels of even dimensions were obtained by a similar method. In [73], a new construction of the fractional Clifford-Fourier kernels was given by solving wave-type problems. In the present chapter the fractional Clifford-Fourier kernel is computed as

$$K_m^p(x, y) = e^{ip\Gamma_y} e^{-i\langle x,y\rangle}.$$

The more general case can also be obtained using our method.

3.2 Laplace transform method

In this section we introduce an auxiliary variable t in the exponent of the classical Fourier transform and then use the Laplace transform to get the Clifford-Fourier kernel in the Laplace domain.

We use the notation $\sqrt{-} := \sqrt{s^2 - |x|^2|y|^2}$. By direct computation, we have $(s + \sqrt{-})(s - \sqrt{-}) = |x|^2|y|^2$ and

$$\begin{aligned}
& \left(1 + \frac{yx}{s+\sqrt{-}}\right)\overline{\left(1 + \frac{yx}{s+\sqrt{-}}\right)} \\
= & \left(1 + \frac{yx}{s+\sqrt{-}}\right)\left(1 + \frac{xy}{s+\sqrt{-}}\right) \\
= & \ 1 + \frac{yx + xy}{s+\sqrt{-}} + \frac{|x|^2|y|^2}{(s+\sqrt{-})^2} \\
= & \ 1 - \frac{2\langle x, y\rangle}{s+\sqrt{-}} + \frac{(s+\sqrt{-})(s-\sqrt{-})}{(s+\sqrt{-})^2} \\
= & \ \frac{2(s - \langle x, y\rangle)}{s+\sqrt{-}}.
\end{aligned} \tag{3.1}$$

Then using (3.1), we can express $\mathcal{L}(t^{m/2-1}e^{t\langle x,y\rangle})$ in terms of the generalized Cauchy kernel introduced in Section 2.2. Fixing $x, y \in \mathbb{R}^m$, assuming $\mathrm{Re}(s) > |x||y|$, we may compute

$$\begin{aligned}
& \mathcal{L}(t^{m/2-1}e^{t\langle x,y\rangle}) \\
= & \ \frac{\Gamma(m/2)}{(s - \langle x, y\rangle)^{m/2}} \\
= & \ \frac{\Gamma(m/2)}{(\frac{s+\sqrt{-}}{2})^{m/2}\left(\left(1 + \frac{yx}{s+\sqrt{-}}\right)\overline{\left(1 + \frac{yx}{s+\sqrt{-}}\right)}\right)^{m/2}} \\
= & \ \frac{\Gamma(m/2)}{(\frac{s+\sqrt{-}}{2})^{m/2}\left(\left(1 + \frac{yx}{s+\sqrt{-}}\right)\left(1 + \frac{xy}{s+\sqrt{-}}\right)\right)^{m/2}} \\
= & \ \frac{\Gamma(m/2)}{(\frac{s+\sqrt{-}}{2})^{m/2}\left(\left(1 + \frac{yx}{s+\sqrt{-}}\right)\left(1 + \frac{xy}{s+\sqrt{-}}\right)\right)^{m/2}} \\
& \times \frac{1 + \frac{yx}{s+\sqrt{-}} - \frac{y(1 + \frac{yx}{s+\sqrt{-}})x}{s+\sqrt{-}}}{\frac{2\sqrt{-}}{s+\sqrt{-}}}
\end{aligned}$$

$$= \frac{2^{m/2-1}\Gamma(m/2)}{\sqrt{-}(s+\sqrt{-})^{m/2-1}} \frac{1+\dfrac{yx}{s+\sqrt{-}} - \dfrac{y(1+\dfrac{yx}{s+\sqrt{-}})x}{s+\sqrt{-}}}{\left(\left(1+\dfrac{yx}{s+\sqrt{-}}\right)\left(1+\dfrac{xy}{s+\sqrt{-}}\right)\right)^{m/2}}. \tag{3.2}$$

The first equality is by (2.4), the second equality by (3.1), and the third equality follows by

$$\begin{aligned} & 1+\frac{yx}{s+\sqrt{-}} - \frac{y(1+\frac{yx}{s+\sqrt{-}})x}{s+\sqrt{-}} \\ = & 1+\frac{yx}{s+\sqrt{-}} - \frac{yx}{s+\sqrt{-}} - \frac{yyxx}{(s+\sqrt{-})^2} \\ = & 1-\frac{(s+\sqrt{-})(s-\sqrt{-})}{(s+\sqrt{-})^2} \\ = & \frac{2\sqrt{-}}{s+\sqrt{-}}. \end{aligned}$$

Next we will compute $\mathcal{L}(t^{m/2-1}e^{ip\Gamma_y}e^{t\langle x,y\rangle})$ by acting with $e^{ip\Gamma_y}$ on both sides of (3.2). The generalized Cauchy kernel $G(y) = \frac{1}{\omega_m}\frac{\bar{y}}{|y|^m}$ is a monogenic function except at the origin. By translation, $\frac{y+x}{|y+x|^m}$ is monogenic in y except at $-x$, and $\frac{y+\frac{1}{x}}{|y+\frac{1}{x}|^m}\frac{x}{|x|^m}$ is monogenic in y except at $-x^{-1}$. The latter function equals $\frac{yx+1}{((yx+1)(\overline{yx+1}))^{m/2}}$. Using Theorem 2.1, for fixed x, we can express $\frac{yx+1}{((yx+1)(\overline{yx+1}))^{m/2}}$ as a series of spherical monogenic polynomials, i.e.

$$\frac{yx+1}{((yx+1)(\overline{yx+1}))^{m/2}} = M_0(y)+M_1(y)+M_2(y)+\cdots$$

where $M_k(y)$ is a spherical monogenic of order k. This series converges uniformly and absolutely for any compact set in $|y| < \frac{1}{|x|}$ (see [12]). Substituting $\frac{y}{s+\sqrt{-}}$ for y, we have

$$\frac{1+\dfrac{yx}{s+\sqrt{-}}}{[(1+\dfrac{yx}{s+\sqrt{-}})(1+\dfrac{xy}{s+\sqrt{-}})]^{m/2}} = \frac{M_0(y)}{(s+\sqrt{-})^0} + \frac{M_1(y)}{(s+\sqrt{-})^1} + \cdots,$$

which converges uniformly and absolutely for any compact set when $\mathrm{Re}(s) > |y||x|$. Using (2.1), we obtain when $\mathrm{Re}(s) > |y||x|$,

$$\begin{aligned}
&\Gamma_y\left(\frac{1+\dfrac{yx}{s+\sqrt{-}}}{[(1+\dfrac{yx}{s+\sqrt{-}})(1+\dfrac{xy}{s+\sqrt{-}})]^{m/2}}\right)\\
= \; & \Gamma_y\left(\sum_{k=0}^{\infty}\frac{M_k(y)}{(s+\sqrt{-})^k}\right)\\
= \; & \sum_{k=0}^{\infty}\frac{\Gamma_y M_k(y)}{(s+\sqrt{-})^k}\\
= \; & \sum_{k=0}^{\infty}\frac{(-k)\cdot M_k(y)}{(s+\sqrt{-})^k}
\end{aligned}$$

because the denominators in the series expansion are radial functions in y and hence commute with Γ_y. This leads to

$$\begin{aligned}
&e^{ip\Gamma_y}\left(\frac{1+\dfrac{yx}{s+\sqrt{-}}}{[(1+\dfrac{yx}{s+\sqrt{-}})(1+\dfrac{xy}{s+\sqrt{-}})]^{m/2}}\right)\\
= \; & \frac{e^{ip\cdot 0}M_0(y)}{(s+\sqrt{-})^0}+\frac{e^{ip\cdot(-1)}M_1(y)}{(s+\sqrt{-})^1}+\frac{e^{ip\cdot(-2)}M_2(y)}{(s+\sqrt{-})^2}+\cdots\\
= \; & M_0\left(\frac{e^{-ip}y}{(s+\sqrt{-})}\right)+M_1\left(\frac{e^{-ip}y}{(s+\sqrt{-})}\right)+M_2\left(\frac{e^{-ip}y}{(s+\sqrt{-})}\right)+\cdots\\
= \; & \frac{1+\dfrac{e^{-ip}yx}{s+\sqrt{-}}}{[(1+\dfrac{e^{-ip}yx}{s+\sqrt{-}})(1+\dfrac{e^{-ip}xy}{s+\sqrt{-}})]^{m/2}}.
\end{aligned}$$

Similarly, by (2.2),

$$e^{ip\Gamma_y}(yM_k(y)) = e^{i(m-2)p+i(k+1)p}(yM_k(y)) = e^{i(m-2)p}(ye^{ip}M_k(e^{ip}y)).$$

Now we can get the desired result

$$\begin{aligned}
&\mathcal{L}(t^{m/2-1}e^{ip\Gamma_y}e^{t\langle x,y\rangle})\\
= \; & \frac{2^{m/2-1}(m/2-1)!}{\sqrt{-}(s+\sqrt{-})^{m/2-1}}\left(\frac{1+\dfrac{e^{-ip}yx}{s+\sqrt{-}}}{[(1+\dfrac{e^{-ip}yx}{s+\sqrt{-}})(1+\dfrac{e^{-ip}xy}{s+\sqrt{-}})]^{m/2}}\right.
\end{aligned}$$

$$-e^{i(m-2)p}\frac{\frac{e^{ip}y(1+\frac{e^{ip}yx}{s+\sqrt{-}})x}{s+\sqrt{-}}}{[(1+\frac{e^{ip}yx}{s+\sqrt{-}})(1+\frac{e^{ip}xy}{s+\sqrt{-}})]^{m/2}}\Bigg).$$

In order to simplify the expression further, we need the following,

$$\begin{aligned}
& [(1+\frac{e^{-ip}yx}{s+\sqrt{-}})(1+\frac{e^{-ip}xy}{s+\sqrt{-}})]^{m/2} \\
= \ & [1-\frac{2e^{-ip}\langle x,y\rangle}{s+\sqrt{-}}+\frac{e^{-2ip}yxxy(s-\sqrt{-})}{(s+\sqrt{-})^2(s-\sqrt{-})}]^{m/2} \\
= \ & [\frac{2e^{-ip}}{s+\sqrt{-}}(1/2e^{ip}(s+\sqrt{-})-\langle x,y\rangle+1/2e^{-ip}(s-\sqrt{-}))]^{\frac{m}{2}} \\
= \ & [\frac{2e^{-ip}}{s+\sqrt{-}}(s\cos p-\langle x,y\rangle+i\sqrt{-}\sin p)]^{\frac{m}{2}},
\end{aligned} \tag{3.3}$$

as well as

$$1+\frac{e^{-ip}yx}{s+\sqrt{-}}=\frac{s+\sqrt{-}+e^{-ip}yx}{s+\sqrt{-}} \tag{3.4}$$

and

$$\frac{y(1+\frac{e^{ip}yx}{s+\sqrt{-}})x}{s+\sqrt{-}}=\frac{yx+e^{ip}(s-\sqrt{-})}{s+\sqrt{-}}=\frac{e^{ip}(e^{-ip}yx+s-\sqrt{-})}{s+\sqrt{-}}. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we get the following theorem.

Theorem 3.1. *For $x, y \in \mathbb{R}^m$, the Laplace transform of the fractional Clifford-Fourier kernel is given by:*

$$\begin{aligned}
& \mathcal{L}(t^{m/2-1}e^{ip\Gamma_y}e^{t\langle x,y\rangle}) \\
= \ & \frac{\Gamma(m/2)}{2\sqrt{-}}\Bigg(\frac{s+\sqrt{-}+e^{-ip}yx}{(e^{-ip}(s\cos p+i\sqrt{-}\sin p-\langle x,y\rangle))^{m/2}} \\
& -e^{imp}\frac{s-\sqrt{-}+e^{-ip}yx}{(e^{ip}(s\cos p-i\sqrt{-}\sin p-\langle x,y\rangle))^{m/2}}\Bigg),
\end{aligned}$$

when $Re(s) > |y||x|$ and here $\sqrt{-}=\sqrt{s^2-|x|^2|y|^2}$.

If we redo the computation with x replaced by $-ix$, similar results can be obtained except that the existence condition $\mathrm{Re}(s) > |x||y|$ is relaxed to $\mathrm{Re}(s) > 0$. The result is collected in the following theorem.

Theorem 3.2. *For $x, y \in \mathbb{R}^m$, the Laplace transform of the fractional Clifford-Fourier kernel is given by:*

$$
\begin{aligned}
& \mathcal{L}(t^{m/2-1} e^{ip\Gamma_y} e^{-it\langle x,y\rangle}) \\
= \; & \frac{\Gamma(m/2)}{2\sqrt{+}} \Bigg(\frac{s + \sqrt{+} - ie^{-ip} yx}{(e^{-ip}(s\cos p + i\sqrt{+}\sin p + i\langle x,y\rangle))^{m/2}} \\
& -e^{imp} \frac{s - \sqrt{+} - ie^{-ip} yx}{(e^{ip}(s\cos p - i\sqrt{+}\sin p + i\langle x,y\rangle))^{m/2}} \Bigg),
\end{aligned}
$$

when $Re(s) > 0$ and with $\sqrt{+} = \sqrt{s^2 + |x|^2|y|^2}$.

3.3 Plane wave decomposition of the kernel

In this section, for $x, y \in \mathbb{R}^m$, we use the notation $\hat{x} = \frac{x}{|x|}$, $\hat{y} = \frac{y}{|y|}$ to denote two unit vectors. For $\hat{x}$, $\hat{y}$ we also have the result in Theorem 3.2. This time, we could get the kernel by putting $t = |x||y|$. Denote $r = \sqrt{s^2 + 1}$, $R = s + \sqrt{s^2 + 1}$, and $\langle \hat{x}, \hat{y}\rangle = \cos\theta$. Using $s = \frac{R - 1/R}{2}$ and $\sqrt{s^2 + 1} = \frac{R + 1/R}{2}$, Theorem 3.2 becomes

$$
\begin{aligned}
& \mathcal{L}(t^{m/2-1} e^{ip\Gamma_y} e^{-it\langle \hat{x},\hat{y}\rangle}) \\
= \; & \frac{\Gamma(m/2)}{r} 2^{m/2-1} R^{-m/2} \Bigg(\frac{R - ie^{-ip}\hat{y}\hat{x}}{(1 + 2\frac{ie^{-ip}}{R}\cos\theta - (\frac{e^{-ip}}{R})^2)^{m/2}} \\
& +e^{imp} \frac{\frac{1}{R} + ie^{-ip}\hat{y}\hat{x}}{(1 + 2\frac{ie^{ip}}{R}\cos\theta - (\frac{e^{ip}}{R})^2)^{m/2}} \Bigg) \\
= \; & \frac{\Gamma(m/2)}{r} 2^{m/2-1} R^{-m/2} \\
& \times \Bigg(\frac{1}{(1 + 2\frac{ie^{-ip}}{R}\cos\theta - (\frac{e^{-ip}}{R})^2)^{m/2}} [ie^{-ip}(-\cos\theta + \frac{-ie^{-ip}}{R}) \\
& +(R + 2ie^{-ip}\cos\theta - \frac{e^{-2ip}}{R}) + ie^{-ip}\hat{x}\wedge\hat{y}] \\
& +e^{imp} \frac{\frac{1}{R} + ie^{-ip}(-\cos\theta + \hat{y}\wedge\hat{x})}{(1 + 2\frac{ie^{ip}}{R}\cos\theta - (\frac{e^{ip}}{R})^2)^{m/2}} \Bigg)
\end{aligned}
\tag{3.6}
$$

With the help of the generating function of the Gegenbauer polynomial [92]

$$
\frac{1}{(1 - 2xt + t^2)^\lambda} = \sum_{k=0}^{\infty} C_k^{(\lambda)}(x) t^k, \tag{3.7}
$$

and its derivative with respect to t

$$-\lambda\frac{-2x+2t}{(1-2xt+t^2)^{\lambda+1}}=\sum_{k=0}^{\infty}kC_k^{(\lambda)}(x)t^{k-1}, \tag{3.8}$$

we can express the right hand side of (3.6) into series. We have by (3.8)

$$\begin{aligned}
&\frac{ie^{-ip}(-\cos\theta+\frac{-ie^{-ip}}{R})}{(1+2\frac{ie^{-ip}}{R}\cos\theta-(\frac{e^{-ip}}{R})^2)^{m/2}}\\
=\;&\frac{-ie^{-ip}}{(m-2)}\sum_{k=0}^{\infty}kC_k^{(\frac{m}{2}-1)}(\cos\theta)(\frac{-ie^{-ip}}{R})^k,
\end{aligned}$$

and by (3.7)

$$\begin{aligned}
&\frac{R+2ie^{-ip}\cos\theta-\frac{e^{-2ip}}{R}}{(1+2\frac{ie^{-ip}}{R}\cos\theta-(\frac{e^{-ip}}{R})^2)^{m/2}}\\
=\;&R\sum_{k=0}^{\infty}C_k^{(\frac{m}{2}-1)}(\cos\theta)(\frac{-ie^{-ip}}{R})^k,
\end{aligned}$$

as well as by (3.7)

$$\begin{aligned}
&\frac{ie^{-ip}\hat{x}\wedge\hat{y}}{(1+2\frac{ie^{-ip}}{R}\cos\theta-(\frac{e^{-ip}}{R})^2)^{m/2}}\\
=\;&ie^{-ip}\hat{x}\wedge\hat{y}\sum_{k=0}^{\infty}C_k^{(\frac{m}{2})}(\cos\theta)(\frac{-ie^{-ip}}{R})^k.
\end{aligned}$$

Similarly, we can get the series expression of the remaining part of (3.6),

$$\begin{aligned}
&e^{imp}\frac{\frac{1}{R}+ie^{-ip}(-\cos\theta+\hat{y}\wedge\hat{x})}{(1+2\frac{ie^{ip}}{R}\cos\theta-(\frac{e^{ip}}{R})^2)^{m/2}}\Bigg)\\
=\;&ie^{i(m-1)p}\Bigg(\hat{y}\wedge\hat{x}\sum_{k=0}^{\infty}C_k^{(\frac{m}{2})}(\cos\theta)(\frac{-ie^{ip}}{R})^k\\
&-\frac{1}{m-2}\sum_{k=0}^{\infty}kC_k^{(\frac{m}{2}-1)}(\cos\theta)(\frac{-ie^{ip}}{R})^k\Bigg).
\end{aligned}$$

Collecting all we got, we have the series expression in the Laplace domain as

$$\mathcal{L}(t^{m/2-1}e^{ip\Gamma_y}e^{-it(\hat{x},\hat{y})}) = \hat{A}_m^p + \hat{B}_m^p + \hat{x}\wedge\hat{y}\hat{C}_m^p$$

where

$$\begin{aligned}
\hat{A}_m^p &= -2^{\frac{m}{2}-2}\frac{\Gamma(m/2)}{r}\sum_{k=0}^{\infty} i^{-k}(e^{ip(k+m-2)} - e^{-ipk}) \\
&\quad C_k^{(\frac{m}{2}-1)}(\cos\theta)\frac{1}{R^{k+\frac{m}{2}-1}}; \\
\hat{B}_m^p &= 2^{\frac{m}{2}-2}\frac{\Gamma(m/2-1)}{r}\sum_{k=0}^{\infty}(k+\frac{m}{2}-1)i^{-k}(e^{ip(k+m-2)} + e^{-ipk}) \\
&\quad C_k^{(\frac{m}{2}-1)}(\cos\theta)\frac{1}{R^{k+\frac{m}{2}-1}}; \\
\hat{C}_m^p &= 2^{\frac{m}{2}-1}\Gamma(m/2)\frac{\hat{x}\wedge\hat{y}}{r}\sum_{k=1}^{\infty} i^{-k}(e^{ip(k+m-2)} - e^{-ipk}) \\
&\quad C_{k-1}^{(\frac{m}{2})}(\cos\theta)\frac{1}{R^{k+\frac{m}{2}-1}}.
\end{aligned}$$

When transforming back by (2.7), we get the plane wave decomposition of the fractional Clifford-Fourier kernel as follows which can be compared with Theorem 3.2 in [26].

Theorem 3.3. *The series representation of the fractional Clifford-Fourier kernel is given by*

$$\begin{aligned}
K_m^p\langle x,y\rangle &= e^{ip\Gamma_y}e^{-i\langle x,y\rangle} \\
&= A_m^p + B_m^p + x\wedge yC_m^p,
\end{aligned}$$

where

$$\begin{aligned}
A_m^p &= -2^{m/2-2}\Gamma(m/2)\sum_{k=0}^{\infty} i^{-k}(e^{ip(k+m-2)} - e^{-ipk})(|x||y|)^{-m/2+1} \\
&\quad J_{m/2-1+k}(|x||y|)C_k^{(m/2-1)}(\cos\theta), \\
B_m^p &= 2^{m/2-2}\Gamma(m/2-1)\sum_{k=0}^{\infty} i^{-k}(k+m/2-1)(e^{ip(k+m-2)} + e^{-ipk}) \\
&\quad (|x||y|)^{-m/2+1}J_{m/2-1+k}(|x||y|)C_k^{(m/2-1)}(\cos\theta),
\end{aligned}$$

$$\begin{aligned} C_m^p &= 2^{m/2-1}\Gamma(m/2)\sum_{k=1}^{\infty} i^{-k}(e^{ip(k+m-2)} - e^{-ipk})(|x||y|)^{-m/2} \\ &\quad J_{m/2-1+k}(|x||y|)C_{k-1}^{(m/2)}(\cos\theta). \end{aligned}$$

Alternatively, using the generating function of the Gegenbauer polynomials, we have

$$\begin{aligned} &(1+2\cos\theta\frac{ie^{-ip}}{R}+(\frac{-ie^{-ip}}{R})^2)^{-m/2} \\ = &\sum_{k=0}^{\infty}(\frac{-ie^{-ip}}{R})^k C_k^{(m/2)}(\cos\theta) \\ = &\sum_{k=0}^{\infty}\sum_{a=0}^{k}(\frac{-ie^{-ip}}{R})^k\frac{(m/2)_a(m/2)_{k-a}}{a!(k-a)!}\cos(k-2a)\theta \end{aligned}$$

which means we can express formula (3.6) equally as a Fourier series.

3.4 Even dimensional Clifford-Fourier kernel

When $p=\pi/2$, the result in Theorem 3.2 reduces to

$$\begin{aligned} &\mathcal{L}(t^{m/2-1}e^{i\frac{\pi}{2}\Gamma_y}e^{-it\langle x,y\rangle}) \\ = &\frac{\Gamma(m/2)}{2\sqrt{+}}\left(\frac{s+\sqrt{+}-yx}{(\sqrt{+}+\langle x,y\rangle)^{m/2}} - e^{im\pi/2}\frac{s-\sqrt{+}-yx}{(\sqrt{+}-\langle x,y\rangle)^{m/2}}\right) \\ = &\frac{\Gamma(m/2)}{2\sqrt{+}}\left(\frac{(s-yx+\sqrt{+})(\sqrt{+}-\langle x,y\rangle)^{m/2}}{(s^2+(ix\wedge y)^2)^{m/2}}\right. \\ &\left.-\frac{e^{im\pi/2}(\sqrt{+}+\langle x,y\rangle)^{m/2}(s-yx-\sqrt{+})}{(s^2+(ix\wedge y)^2)^{m/2}}\right). \end{aligned} \tag{3.9}$$

When $m/2$ is even, (3.9) becomes

$$\begin{aligned} &(m/2-1)!\left(\frac{(s-yx)(\sum_{j=1,3,5,\cdots}\binom{\frac{m}{2}}{j}(\sqrt{+})^{m/2-j-1}(-\langle x,y\rangle)^j)}{(s^2+(ix\wedge y)^2)^{m/2}}\right. \\ &\left.+\frac{\sum_{j=0,2,4,\cdots}\binom{\frac{m}{2}}{j}(\sqrt{+})^{m/2-j}(-\langle x,y\rangle)^j}{(s^2+(ix\wedge y)^2)^{m/2}}\right) \\ = &(m/2-1)!\Big((s-yx) \end{aligned}$$

$$
\begin{aligned}
&\times\frac{\left(\sum_{j=1,3,5,\cdots}\binom{\frac{m}{2}}{j}(s^2+(ix\wedge y)^2+\langle x,y\rangle^2)^{\frac{m/2-j-1}{2}}(-\langle x,y\rangle)^j\right)}{(s^2+|x\wedge y|^2)^{m/2}}\\
&+\frac{\sum_{j=0,2,4,\cdots}\binom{\frac{m}{2}}{j}(\sqrt{+})^{m/2-j}(-\langle x,y\rangle)^j}{(s^2+|x\wedge y|^2)^{m/2}}\Bigg)\\
= &\ (m/2-1)!\Bigg((s-yx)\\
&\times\frac{\left(\sum_{j=0}^{m/4-1}\binom{\frac{m}{2}}{2j+1}[(s^2+|x\wedge y|^2)+\langle x,y\rangle^2]^{\frac{m/2-2j-2}{2}}(-\langle x,y\rangle)^{2j+1}\right)}{(s^2+|x\wedge y|^2)^{m/2}}\\
&+\frac{\sum_{k=0}^{m/4}\binom{\frac{m}{2}}{2k}[(s^2+|x\wedge y|^2)+\langle x,y\rangle^2]^{\frac{m/2-2k}{2}}(-\langle x,y\rangle)^{2k}}{(s^2+|x\wedge y|^2)^{m/2}}\Bigg) \qquad (3.10)
\end{aligned}
$$

where all the sums are finite. By the binomial theorem, we have

$$
\begin{aligned}
&[(s^2+|x\wedge y|^2)+\langle x,y\rangle^2]^{\frac{m/2-j-1}{2}}\\
=&\sum_{k=0}^{\frac{m/2-j-1}{2}}\binom{\frac{m/2-j-1}{2}}{k}(s^2+|x\wedge y|^2)^k\langle x,y\rangle^{m/2-j-1-2k},
\end{aligned}
$$

Formula (3.10) becomes

$$
\begin{aligned}
&(m/2-1)!\Bigg((s-yx)\\
&\times\sum_{j=1,3,5,\cdots,m/2-1}\binom{\frac{m}{2}}{j}\sum_{k=0}^{\frac{m/2-j-1}{2}}\binom{\frac{m/2-j-1}{2}}{k}\frac{(-1)^j\langle x,y\rangle^{m/2-1-2k}}{(s^2+|x\wedge y|^2)^{m/2-k}}\\
&+\sum_{u=0,2,4,\cdots,m/2}\binom{\frac{m}{2}}{u}\sum_{v=0}^{\frac{m/2-u-1}{2}}\binom{\frac{m/2-u-1}{2}}{v}\frac{(-1)^u\langle x,y\rangle^{m/2-1-2v}}{(s^2+|x\wedge y|^2)^{m/2-v}}\Bigg)
\end{aligned}
\qquad (3.11)
$$

which is a finite sum of rational functions of type $\frac{\langle x,y\rangle^k}{(s^2+|x\wedge y|^2)^q}$, $\frac{s\langle x,y\rangle^k}{(s^2+|x\wedge y|^2)^q}$ and $y\wedge x\frac{s\langle x,y\rangle^k}{(s^2+|x\wedge y|^2)^q}$. Formulas (2.8) and (2.9) show that the kernel can be expressed as a finite sum of Bessel functions. Now we can get the kernel expressed in terms of Bessel functions which has been obtained in a completely different way in [29].

Theorem 3.4. *The kernel of the Clifford-Fourier transform for even dimension $m = 4n, n \geq 1$ is given by*

$$\begin{aligned} K_m(x,y) &= e^{i\frac{\pi}{2}\Gamma_y} e^{-i\langle x,y\rangle} \\ &= (\pi/2)^{1/2}\Big(A_m(u,v) + B_m(u,v) + (x \wedge y) C_m(u,v)\Big) \end{aligned}$$

where $u = \langle x, y\rangle$ and $v = |x \wedge y|$ and

$$A_m(u,v) = \sum_{l=0}^{m/4-1} u^{m/2-2-2l} \frac{1}{2^l l!} \frac{\Gamma(m/2)}{\Gamma(m/2-2l-1)} \frac{J_{(m-2l-3)/2}(v)}{v^{(m-2l-3)/2}},$$

$$B_m(u,v) = -\sum_{l=0}^{m/4-1} u^{m/2-1-2l} \frac{1}{2^l l!} \frac{\Gamma(m/2)}{\Gamma(m/2-2l)} \frac{J_{(m-2l-3)/2}(v)}{v^{(m-2l-3)/2}},$$

$$C_m(u,v) = -\sum_{l=0}^{m/4-1} u^{m/2-1-2l} \frac{1}{2^l l!} \frac{\Gamma(m/2)}{\Gamma(m/2-2l)} \frac{J_{(m-2l-1)/2}(v)}{v^{(m-2l-1)/2}}.$$

Similarly, we can get the result when $m/2$ is odd.

We can also obtain an alternative expression using exponentials. When m is even, we have found that formula (3.9) became

$$\mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-ti\langle x,y\rangle}) = \frac{\text{polynomial of } s}{\text{polynomial of } s}.$$

Hence we can use partial fractions to transform back, as

$$\mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-t(ix,y)}) = \sum_{j=1}^{2}\sum_{k=1}^{m/2} \frac{C_{jk}}{(s-\alpha_j)^k} + yx \sum_{p=1}^{2}\sum_{q=1}^{m/2} \frac{\tilde{C}_{pq}}{(s-\alpha_p)^q}.$$

Each $C_{jk}, \tilde{C}_{pq}$ can be obtained by the usual technique of partial fractions.

In particular, the kernel of the 2-dimensional Clifford-Fourier transform can be obtained as follows. Formula (3.9) becomes

$$\frac{1}{2\sqrt{+}} \frac{2(s-yx)\sqrt{+} - 2\sqrt{+}\langle x,y\rangle}{s^2 - (x \wedge y)^2}$$

$$= \quad \frac{s - yx - \langle x, y\rangle}{s^2 - (x \wedge y)^2} = \frac{s + (x \wedge y)}{s^2 - (x \wedge y)^2} = \frac{1}{s - (x \wedge y)}.$$

Transforming back, using (2.3), we get the kernel

$$K_2\langle x, y\rangle = e^{x \wedge y}.$$

This should be compared with section 4.2 in [13] and Proposition 5.1 in [29].

3.5 New integral expressions for the kernels

When $p = \pi/2$, Theorem 3.2 becomes for fixed $x, y \in \mathbb{R}^m$,

$$\begin{aligned} &\mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-ti\langle x,y\rangle}) \\ = \quad &\frac{(m/2-1)!}{2\sqrt{+}} \left(\frac{s + \sqrt{+} - yx}{(\sqrt{+} + \langle x, y\rangle)^{m/2}} - e^{im\pi/2} \frac{s - \sqrt{+} - yx}{(\sqrt{+} - \langle x, y\rangle)^{m/2}} \right) \end{aligned} \tag{3.12}$$

with $\mathrm{Re}(s) > 0$. By (2.10), (2.11) and (2.12), we have

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{s}{\sqrt{+}} \frac{1}{(\sqrt{+} + \langle x, y\rangle)^{m/2}} \right) = \frac{t^{m/2-1}}{(m/2-1)!} e^{-\langle x,y\rangle t} \\ &-|x||y|t \int_0^t (t^2 - u^2)^{-1/2} J_1[|x||y|(t^2 - u^2)^{1/2}] \frac{u^{m/2-1}}{(m/2-1)!} e^{-\langle x,y\rangle u} du, \end{aligned}$$

as well as

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{1}{\sqrt{+}} \frac{1}{(\sqrt{+} + \langle x, y\rangle)^{m/2-1}} \right) \\ = \quad &\int_0^t J_0[|x||y|(t^2 - u^2)^{1/2}] \frac{u^{m/2-2}}{(m/2-2)!} e^{-\langle x,y\rangle u} du, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{1}{\sqrt{+}} \frac{x \wedge y}{(\sqrt{+} + \langle x, y\rangle)^{m/2}} \right) \\ = \quad &\int_0^t (x \wedge y) J_0[|x||y|(t^2 - u^2)^{1/2}] \frac{u^{m/2-1}}{(m/2-1)!} e^{-\langle x,y\rangle u} du. \end{aligned}$$

Using the above three formulas, we can find the result in the time domain. Then setting $t = 1$, we get a new representation of Clifford-Fourier kernel for both even and odd dimension.

Theorem 3.5. *For $x, y \in \mathbb{R}^m, m \geq 3$, the kernel for the m-dimensional Clifford-Fourier transform is given by*

$$
\begin{aligned}
& K_m(x,y) \\
= \quad & \frac{e^{-\langle x,y\rangle}}{2} - \frac{|x||y|}{2}\int_0^1 (1-u^2)^{-1/2} J_1[|x||y|(1-u^2)^{1/2}]u^{m/2-1}e^{-\langle x,y\rangle u}du \\
& + \frac{m-2}{4}\int_0^1 J_0[|x||y|(1-u^2)^{1/2}]u^{m/2-2}e^{-\langle x,y\rangle u}du \\
& + \frac{1}{2}\int_0^1 (x\wedge y)J_0[|x||y|(1-u^2)^{1/2}]u^{m/2-1}e^{-\langle x,y\rangle u}du - e^{im\pi/2}\Bigg(\frac{e^{\langle x,y\rangle}}{2} \\
& - \frac{|x||y|}{2}\int_0^1 (1-u^2)^{-1/2} J_1[|x||y|(1-u^2)^{1/2}]u^{m/2-1}e^{\langle x,y\rangle u}du \\
& - \frac{m-2}{4}\int_0^1 J_0[|x||y|(1-u^2)^{1/2}]u^{m/2-2}e^{\langle x,y\rangle u}du \\
& + \frac{1}{2}\int_0^1 (x\wedge y)J_0[|x||y|(1-u^2)^{1/2}]u^{m/2-1}e^{\langle x,y\rangle u}du\Bigg).
\end{aligned}
$$

Remark 3.1. The 2-dimensional kernel was given in the previous section. In this integral representation, the integral is divergent when $m = 2$.

3.6 Generating function for the even dimensional Clifford-Fourier kernels

In this section we compute the formal generating function of all even dimensional kernels

$$G_p(x,y,a) = \sum_{m=2,4,6,\cdots} \frac{K_m^p(x,y)a^{m/2-1}}{\Gamma(m/2)},$$

where $K_m^p(x,y)$ is the kernel of dimension m. Here the formal generating function means one can obtain the kernel from the derivatives of the generating function. Note that the kernel $K_m^p(x,y)$ is in fact a function of $\langle x,y\rangle$, $|x||y|$ and $|x\wedge y|$. Recall that $\frac{x\wedge y}{|x\wedge y|}$ can be considered as an imaginary unit. So the sum $G_p(x,y,a)$ is not a sum of functions from different spaces but a sum of functions defined on R^3. In order to get the generating function, we introduce a new variable

t in the generating function $G_p(x,y,a)$, i.e.

$$G_p(x,y,a,t) = \sum_{m=2,4,6,\cdots} \frac{1}{(m/2-1)!} t^{m/2-1} e^{ip\Gamma_y} e^{-it\langle x,y\rangle} a^{m/2-1}.$$

It is easy to find that $G_p(x,y,a,1) = G_p(x,y,a)$.

When $p = \pi/2$, the Laplace transform of $G_{\frac{\pi}{2}}(x,y,a,t)$ with respect to t can be computed by

$$\begin{aligned}
& \sum_{m=2,4,6,\cdots} \frac{1}{(m/2-1)!} \mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y} e^{-it\langle x,y\rangle} a^{m/2-1}) \\
= & \frac{1}{2\sqrt{+}} \sum_{m=2,4,6,\cdots} a^{m/2-1} \Bigg(\frac{s+\sqrt{+}-yx}{(\sqrt{+}+\langle x,y\rangle)^{m/2}} \\
& -e^{im\pi/2} \frac{s-\sqrt{+}-yx}{(\sqrt{+}-\langle x,y\rangle)^{m/2}} \Bigg) \\
= & \frac{s+\sqrt{+}-yx}{2\sqrt{+}(\sqrt{+}+\langle x,y\rangle - a)} - e^{i\pi} \frac{s-\sqrt{+}-yx}{2\sqrt{+}(\sqrt{+}-\langle x,y\rangle - ae^{i\pi})} \\
= & \frac{s+\sqrt{+}-yx}{2\sqrt{+}(\sqrt{+}+\langle x,y\rangle - a)} + \frac{s-\sqrt{+}-yx}{2\sqrt{+}(\sqrt{+}-\langle x,y\rangle + a)}, \qquad (3.13)
\end{aligned}$$

the first equality is by (3.12). Using (2.10), (2.11) and (2.12), we then get

$$\begin{aligned}
& \mathcal{L}^{-1}\left(\frac{s+\sqrt{+}-yx}{\sqrt{+}(\sqrt{+}+\langle x,y\rangle - a)} \right) \\
= & 2e^{(-\langle x,y\rangle + a)t} - |x||y|t \int_0^t (t^2-u^2)^{-1/2} J_1[|x||y|(t^2-u^2)^{1/2}] \\
& \times e^{(-\langle x,y\rangle + a)u} du - |x||y| \int_0^t e^{(-\langle x,y\rangle + a)(t^2-u^2)^{1/2}} J_1(|x||y|u) du \\
- & yx \int_0^t J_0[|x||y|(t^2-u^2)^{1/2}] e^{(-\langle x,y\rangle + a)u} du \qquad (3.14)
\end{aligned}$$

as well as

$$\begin{aligned}
& \mathcal{L}^{-1}\left(\frac{s-\sqrt{+}-yx}{\sqrt{+}(\sqrt{+}-\langle x,y\rangle + a)} \right) \\
= & -|x||y|t \int_0^t (t^2-u^2)^{-1/2} J_1[|x||y|(t^2-u^2)^{1/2}] e^{(\langle x,y\rangle - a)u} du
\end{aligned}$$

$$
\begin{aligned}
&+|x||y|\int_0^t e^{(\langle x,y\rangle-a)(t^2-u^2)^{1/2}}J_1(|x||y|u)du\\
&+yx\int_0^t J_0[|x||y|(t^2-u^2)^{1/2}]e^{(\langle x,y\rangle-a)u}du. \qquad (3.15)
\end{aligned}
$$

Combining (3.14) and (3.15), the generating function is

$$
\begin{aligned}
&G_{\pi/2}(x,y,a)=e^{(-\langle x,y\rangle+a)}-|x||y|\\
&\times\int_0^1(1-u^2)^{-1/2}J_1[|x||y|(1^2-u^2)^{1/2}]\cosh((\langle x,y\rangle-a)u)du\\
&+|x||y|\int_0^1\sinh[(\langle x,y\rangle-a)(1-u^2)^{1/2}]J_1(|x||y|u)du\\
&+yx\int_0^1 J_0[|x||y|(1-u^2)^{1/2}]\sinh[(\langle x,y\rangle-a)u]du
\end{aligned}
$$

which only gives an integral representation. In the following, we will use different inverse transform techniques to get the closed form. Simplifying (3.13) further, we have

$$
\begin{aligned}
&\sum_{m=2,4,6,\cdots}\frac{1}{(m/2-1)!}\mathcal{L}(t^{m/2-1}e^{i\frac{\pi}{2}\Gamma_y}e^{-it\langle x,y\rangle}a^{m/2-1})\\
=\;&\frac{s+\sqrt{+}-yx}{2\sqrt{+}(\sqrt{+}+\langle x,y\rangle-a)}+\frac{s-\sqrt{+}-yx}{2\sqrt{+}(\sqrt{+}-\langle x,y\rangle+a)}\\
=\;&\frac{s-yx-\langle x,y\rangle+a}{s^2+|x|^2|y|^2-(\langle x,y\rangle-a)^2}.
\end{aligned}
$$

Transforming back, we get

$$
\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{s-yx-\langle x,y\rangle+a}{s^2+|x|^2|y|^2-(\langle x,y\rangle-a)^2}\right)\\
=\;&\cos(\sqrt{|x|^2|y|^2-(\langle x,y\rangle-a)^2}t)\\
&+\frac{-yx-\langle x,y\rangle+a}{\sqrt{|x|^2|y|^2-(\langle x,y\rangle-a)^2}}\sin(\sqrt{|x|^2|y|^2-(\langle x,y\rangle-a)^2}t).
\end{aligned}
$$

The last equality is by (2.5) and (2.6). Note that it equals the case $m=2$ when $a=0$.

Alternatively, a tedious computation shows that

$$
\frac{s-yx-\langle x,y\rangle+a}{s^2+|x|^2|y|^2-(\langle x,y\rangle-a)^2}=\begin{pmatrix}0 & 1\end{pmatrix}\left(sI+A\right)^{-1}\begin{pmatrix}1\\1\end{pmatrix}
$$

with I the 2×2 identity matrix and A given by

$$A=\left((a-\langle x,y\rangle)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\begin{pmatrix}0 & -xy\\ yx & 0\end{pmatrix}\right)=\begin{pmatrix}a-\langle x,y\rangle & -xy\\ yx & -a+\langle x,y\rangle\end{pmatrix}.$$

We get the following

Theorem 3.6. *The generating function for even dimensional Clifford-Fourier kernels for $p=\frac{\pi}{2}$ is given by*

$$\begin{aligned}G_{\pi/2}(x,y,a) &= \begin{pmatrix}0 & 1\end{pmatrix} e^{-A}\begin{pmatrix}1\\ 1\end{pmatrix}\\ &= \cos(\sqrt{|x|^2|y|^2-(\langle x,y\rangle-a)^2})\\ &\quad +(-yx-\langle x,y\rangle+a)\frac{\sin(\sqrt{|x|^2|y|^2-(\langle x,y\rangle-a)^2})}{\sqrt{|x|^2|y|^2-(\langle x,y\rangle-a)^2}}.\end{aligned}$$

We can get a similar result for the fractional case, i.e. general p. Denote

$$G_p(x,y,a)=\sum_{m=2,4,6,\cdots}\frac{K_m^p(x,y)a^{m/2-1}}{(m/2-1)!}.$$

Now

$$\begin{aligned}&\sum_{m=2,4,\cdots}\mathcal{L}\left(\frac{t^{m/2-1}e^{ip\Gamma_y}e^{-it\langle x,y\rangle}a^{m/2-1}}{(m/2-1)!}\right)\\ =\ & \frac{e^{ip}}{2\sqrt{+}}\left(\frac{s+\sqrt{+}-ie^{-ip}yx}{s\cos p+i\sqrt{+}\sin p+i\langle x,y\rangle-ae^{ip}}\right.\\ &\left.-\frac{s-\sqrt{+}-ie^{-ip}yx}{s\cos p-i\sqrt{+}\sin p+i\langle x,y\rangle-ae^{ip}}\right)\\ =\ & e^{ip}\frac{(-is-e^{-ip}yx)\sin p+(s\cos p+i\langle x,y\rangle-ae^{ip})}{(s\cos p+i\langle x,y\rangle-ae^{ip})^2+(\sqrt{+})^2\sin^2 p}\\ =\ & \frac{s-yx\sin p+i\langle x,y\rangle e^{ip}-ae^{2ip}}{(s\cos p+i\langle x,y\rangle-ae^{ip})^2+(\sqrt{+})^2\sin^2 p},\end{aligned}$$

transforming back by (2.5) and (2.6), we have

$$\begin{aligned}&\mathcal{L}^{-1}\left(\frac{s-yx\sin p+i\langle x,y\rangle e^{ip}-ae^{2ip}}{(s\cos p+i\langle x,y\rangle-ae^{ip})^2+(\sqrt{+})^2\sin^2 p}\right)\\ =\ & e^{-ct}(\cos(dt)+\frac{(x\wedge y-iae^{ip})\sin p}{d}\sin(dt))\end{aligned}$$

with $c = (i\langle x, y\rangle - ae^{ip})\cos p$, $d = \sin p\sqrt{|x|^2|y|^2 + (i\langle x, y\rangle - ae^{ip})^2}$.

Alternatively, we have

$$\begin{aligned} & \frac{s - yx\sin p + i\langle x, y\rangle e^{ip} - ae^{2ip}}{(s\cos p + i\langle x, y\rangle - ae^{ip})^2 + (\sqrt{+})^2\sin^2 p} \\ = & \begin{pmatrix} 0 & 1 \end{pmatrix} \Big(sI + B\Big)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned}$$

where

$$B = \begin{pmatrix} -\beta_+ & 0 \\ -(-yx\sin p + i\langle x, y\rangle e^{ip} - ae^{2ip} + \beta_+) & -\beta_- \end{pmatrix},$$

with $\beta_\pm$ the roots of $(s\cos p + i\langle x, y\rangle - ae^{ip})^2 + (\sqrt{+})^2\sin^2 p$, i.e.

$$\beta_\pm = (-i\langle x, y\rangle + ae^{ip})\cos p \pm \sin p(\sqrt{-|x|^2|y|^2 - (i\langle x, y\rangle - ae^{ip})^2}).$$

Again, we have

Theorem 3.7. *The generating function for the even dimensional fractional Clifford-Fourier kernels is given by*

$$\begin{aligned} G_p(x, y, a) &= \begin{pmatrix} 0 & 1 \end{pmatrix} e^{-B} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^{-c}(\cos d + \frac{(x \wedge y - iae^{ip})\sin p}{d}\sin d) \end{aligned}$$

with $c = (i\langle x, y\rangle - ae^{ip})\cos p$ *and* $d = \sin p\sqrt{|x|^2|y|^2 + (i\langle x, y\rangle - ae^{ip})^2}$.

At the end of this section, we give the kernel for general p when $m = 2$. It corresponds to the case when $a = 0$ in the generating function. The kernel for dimension 2 is hence given by

$$K_2^p(x, y) = e^{-i\langle x, y\rangle\cos p}e^{x\wedge y\sin p},$$

which coincides with the work in [73] and [26].

It is wrong always, everywhere, and for anyone, to believe anything upon insufficient evidence.

William K. Clifford

4
Generalized Clifford-Fourier kernel

According to investigations in [28] using the representation theory for the Lie superalgebra $\mathfrak{osp}(1|2)$, the following expression

$$e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i\langle x,y\rangle} \tag{4.1}$$

where G is an integer-valued polynomial can be used as the kernel for a generalized Fourier transform that still satisfies properties very close to that of the classical transform.

Our aim in the present chapter is to develop further the Laplace transform method for this much wider class of generalized Fourier transforms. The extension of the Laplace transform technique to kernels of type (4.1) will allow us to find explicit expressions for the kernel. We will moreover determine which polynomials G give rise to polynomially bounded kernels and we will determine the generating function corresponding to a fixed polynomial G.

This chapter is organized as follows. In Section 4.1, we give the basis facts concerning the generalized Clifford-Fourier transform. The remainder is devoted to establishing the connection between the kernel of the fractional Clifford-Fourier transform in Chapter 3 and the generalized Clifford-Fourier transform. We first compute a special case in Section 4.2. Then the method is generalized to the case in which the polynomial has integer coefficients in Section 4.3. The kernel and the generating function in the even dimensional case are given. We also discuss which kernels are polynomially bounded.

4.1 The generalized Clifford-Fourier transform

The Clifford-Hermite functions are given by

$$\psi_{2p,k,l}(x) := 2^p p! L_p^{\frac{m}{2}+k-1}(|x|^2) M_k^l e^{-|x|^2/2},$$

$$\psi_{2p+1,k,l}(x) := 2^p p! L_p^{\frac{m}{2}+k}(|x|^2) x M_k^l e^{-|x|^2/2},$$

where $p, k \in \mathbb{Z}_{\geq 0}$ and $\{M_k^l | l = 1, \ldots, \dim(\mathcal{M}_k)\}$ form a basis for $\mathcal{M}_k$, the space of spherical monogenics of degree k [91]. They moreover realize the complete decomposition of the rapidly decreasing functions $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_m \subset L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_m$ in irreducible subspaces under the action of the dual pair $(Spin(m), \mathfrak{osp}(1|2))$. Here the Lie superalgebra $\mathfrak{osp}(1|2)$ is realized by the Dirac operator D and the vector x.

Recall that the Gamma operator or the angular Dirac operator (see [12]) is defined by

$$\Gamma_x := -\sum_{j<k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j}) = -x D_x - \mathbb{E}_x = -x \wedge D_x, \quad (4.2)$$

here $\mathbb{E}_x = \sum_{i=1}^m x_i \partial_{x_i}$ is the Euler operator. The Scasimir S in our operator realization of $\mathfrak{osp}(1|2)$ is related to the angular Dirac operator by $S = -\Gamma_x + \frac{m-1}{2}$, see [50]. The Casimir element $C = S^2$ acts on the Clifford-Hermite function by

$$C \psi_{j,k,l} = \left(k + \frac{m-1}{2}\right)^2 \psi_{j,k,l}.$$

In [28], the authors studied the full class of integral transforms which satisfy the conditions stated in the following theorem.

Theorem 4.1. *The properties*

(1) the Clifford-Helmholtz relations

$$T \circ D_x = -iy \circ T,$$

$$T \circ x = -i D_y \circ T,$$

(2) $T\psi_{j,k,l} = \mu_{j,k} \psi_{j,k,l}$ *with* $\mu_{j,k} \in \mathbb{C}$,

(3) $T^4 = id$
are satisfied by the operators T of the form

$$T = e^{i\frac{\pi}{2}F(C)}e^{i\frac{\pi}{4}(\Delta-|x|^2-m)} \in e^{i\frac{\pi}{2}\bar{\mathcal{U}}(\mathfrak{osp}(1|2)))}$$

where $F(C)$ is a power series in C that takes integer values when evaluated in the eigenvalues of C and $\bar{\mathcal{U}}(\mathfrak{osp}(1|2)$ is the extension of the universal enveloping algebra that allows infinite power series in the elements of $\mathfrak{sl}_2$.

The integral kernel of the generalized Fourier transform T can be expressed as $e^{i\frac{\pi}{2}F(C)}e^{-i\langle x,y\rangle}$. We are in particular interested in the case where $F(C)$ reduces to a polynomial $G(\Gamma_y)$ with integer coefficients.

Remark 4.1. In general, when $G(x) \neq 0$, the generalized Fourier transform T and the Clifford fractional Fourier transform in [11] are two different classes of transforms because their eigenvalues on the Clifford-Hermite functions are different.

Remark 4.2. The Clifford Fourier transform in $C\ell_{(3,0)}$ can also been expressed by operator exponential, see e.g. [46].

4.2 Closed expression for $e^{i\frac{\pi}{2}\Gamma_y^2}e^{-i\langle x,y\rangle}$

In this section, we use the Laplace transform method to compute $e^{i\frac{\pi}{2}\Gamma_y^2}e^{-i\langle x,y\rangle}$. The technique developed here will be used to compute the more general case in next section. We use the notation $\sqrt{+} := \sqrt{s^2+|x|^2|y|^2}$. Using a variant of formula (3.2), we have the following lemma.

Lemma 4.1. *The Laplace transform of $t^{m/2-1}e^{-it\langle x,y\rangle}$ can be expressed as*

$$\begin{aligned}&\mathcal{L}(t^{m/2-1}e^{-it\langle x,y\rangle})\\ =\ & \frac{2^{m/2-1}\Gamma(m/2)}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \frac{1-\dfrac{iyx}{s+\sqrt{+}}+\dfrac{iy(1-\dfrac{iyx}{s+\sqrt{+}})x}{s+\sqrt{+}}}{\left|1-\dfrac{iyx}{s+\sqrt{+}}\right|^m}\end{aligned} \quad (4.3)$$

In the following, we will act with $e^{i\frac{\pi}{2}\Gamma_y^2}$ on both sides of (4.3) to obtain the integral kernel in the Laplace domain. Denote by

$$f(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \frac{1-\dfrac{iyx}{s+\sqrt{+}}}{\left|1-\dfrac{iyx}{s+\sqrt{+}}\right|^m} = \frac{s+\sqrt{+}-iyx}{\sqrt{+}(s+i\langle x,y\rangle)^{m/2}},$$

and

$$\begin{aligned} g(y) &= \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \frac{\dfrac{iy(1-\dfrac{iyx}{s+\sqrt{+}})x}{s+\sqrt{+}}}{\left|1-\dfrac{iyx}{s+\sqrt{+}}\right|^m} \\ &= \frac{iy}{s+\sqrt{+}} f(y)x = \frac{\sqrt{+}-s+iyx}{\sqrt{+}(s+i\langle x,y\rangle)^{m/2}}. \end{aligned}$$

In Section 3.3, it has been proved that $f(y)$ has a series expansion as

$$f(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \sum_{k=0}^{\infty} \frac{M_k(y)}{(s+\sqrt{+})^k}.$$

Here we rewrite

$$f(y) = f_0(y) + f_1(y) + f_2(y) + f_3(y),$$

with

$$f_k(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \sum_{n=0}^{\infty} \frac{M_{4n+k}(y)}{(s+\sqrt{+})^{4n+k}}, \quad k = 0, 1, 2, 3. \quad (4.4)$$

Each f_k is an eigenfunction of the operator $e^{i\frac{\pi}{2}\Gamma^2}$. In fact, by (2.1), we have

$$e^{i\frac{\pi}{2}\Gamma_y^2} M_k(y) = e^{i\frac{\pi}{2}(-k)^2} M_k(y),$$

so

$$\begin{aligned} e^{i\frac{\pi}{2}\Gamma_y^2} M_{4n}(y) &= M_{4n}(y); \\ e^{i\frac{\pi}{2}\Gamma_y^2} M_{4n+1}(y) &= iM_{4n+1}(y); \\ e^{i\frac{\pi}{2}\Gamma_y^2} M_{4n+2}(y) &= M_{4n+2}(y); \end{aligned}$$

$$e^{i\frac{\pi}{2}\Gamma_y^2}M_{4n+3}(y) = iM_{4n+3}(y), \tag{4.5}$$

here $n = 0, 1, 2, \cdots$. Since the operator Γ commutes with radial functions, we know that each f_k is an eigenfunction of $e^{i\frac{\pi}{2}\Gamma^2}$ and the eigenvalues are given in (4.5). In the following, we denote

$$f_\alpha(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}}\sum_{k=0}^{\infty}\frac{M_k(iy)}{(s+\sqrt{+})^k} = \frac{s+\sqrt{+}+yx}{\sqrt{+}(\sqrt{+}-\langle x,y\rangle)^{m/2}},$$

$$f_\beta(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}}\sum_{k=0}^{\infty}\frac{M_k(-y)}{(s+\sqrt{+})^k} = \frac{s+\sqrt{+}+iyx}{\sqrt{+}(s-i\langle x,y\rangle)^{m/2}},$$

$$f_\gamma(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}}\sum_{k=0}^{\infty}\frac{M_k(-iy)}{(s+\sqrt{+})^k} = \frac{s+\sqrt{+}-yx}{\sqrt{+}(\sqrt{+}+\langle x,y\rangle)^{m/2}}$$

as well as

$$g_\alpha(y) = \frac{iy}{s+\sqrt{+}}f_\alpha(y)x = \frac{i(\sqrt{+}-s)+iyx}{\sqrt{+}(\sqrt{+}-\langle x,y\rangle)^{m/2}},$$
$$g_\beta(y) = \frac{iy}{s+\sqrt{+}}f_\beta(y)x = \frac{s-\sqrt{+}+iyx}{\sqrt{+}(s-i\langle x,y\rangle)^{m/2}},$$
$$g_\gamma(y) = \frac{iy}{s+\sqrt{+}}f_\gamma(y)x = \frac{i(s-\sqrt{+})+iyx}{\sqrt{+}(\sqrt{+}+\langle x,y\rangle)^{m/2}}.$$

Remark 4.3. Comparing with the result in Section 3.2, $\frac{\Gamma(m/2)}{2}(f_\gamma + g_\alpha)$ is the Clifford-Fourier kernel of dimension $m = 4n+1, n \in \mathbb{N}$ in the Laplace domain. Denote the first part of the fractional Clifford-Fourier kernel as

$$F_p(x,y) = \frac{s+\sqrt{+}-ie^{-ip}yx}{\sqrt{+}(e^{-ip}(s\cos p + i\sqrt{+}\sin p + i\langle x,y\rangle))^{m/2}}$$

and the second part of the kernel as

$$G_p(x,y) = -e^{ip}\frac{s-\sqrt{+}-ie^{-ip}yx}{\sqrt{+}(e^{ip}(s\cos p - i\sqrt{+}\sin p + i\langle x,y\rangle))^{m/2}}.$$

We find that $f(y) = F_0(x,y)$, $f_\alpha(y) = F_{-\frac{\pi}{2}}(x,y)$, $f_\beta(y) = F_\pi(x,y)$, $f_\gamma(y) = F_{\frac{\pi}{2}}(x,y)$, $g(y) = G_0(x,y)$, $g_\alpha(y) = G_{\frac{\pi}{2}}(x,y)$, $g_\beta(y) = G_\pi(x,y)$ and $g_\gamma(y) = G_{-\frac{\pi}{2}}(x,y)$. We could get the plane wave expansion and integral expression of $f, f_\alpha, f_\beta, f_\gamma$ and $g, g_\alpha, g_\beta, g_\gamma$ from Section 3.3.

As M_k is a polynomial of degree k, we have the following relations,

$$\begin{cases} f(y) = f_0(y) + f_1(y) + f_2(y) + f_3(y); \\ f_\alpha(y) = f_0(y) + if_1(y) - f_2(y) - if_3(y); \\ f_\beta(y) = f_0(y) - f_1(y) + f_2(y) - f_3(y); \\ f_\gamma(y) = f_0(y) - if_1(y) - f_2(y) + if_3(y). \end{cases}$$

Each $f_k(y)$ can be obtained as follows:

$$\begin{cases} 4f_0(y) = f(y) + f_\alpha(y) + f_\beta(y) + f_\gamma(y); \\ 4f_1(y) = f(y) - if_\alpha(y) - f_\beta(y) + if_\gamma(y); \\ 4f_2(y) = f(y) - f_\alpha(y) + f_\beta(y) - f_\gamma(y); \\ 4f_3(y) = f(y) + if_\alpha(y) - f_\beta(y) - if_\gamma(y). \end{cases} \tag{4.6}$$

Now the action of $e^{i\frac{\pi}{2}\Gamma_y^2}$ on $f(y)$ is known through its eigenfunctions,

$$\begin{aligned} e^{i\frac{\pi}{2}\Gamma_y^2} f(y) &= e^{i\frac{\pi}{2}\Gamma_y^2}\Big(f_0(y) + f_1(y) + f_2(y) + f_3(y)\Big) \\ &= f_0(y) + if_1(y) + f_2(y) + if_3(y) \\ &= \frac{1}{2}\Big(f(y) + f_\beta(y) + if(y) - if_\beta(y)\Big). \end{aligned}$$

The case $e^{i\frac{\pi}{2}\Gamma_y^2} g(y)$ can be treated similarly, using (4.5) and

$$\begin{aligned} e^{i\frac{\pi}{2}\Gamma_y^2}(yM_k(y)) &= e^{i\frac{\pi}{2}(m-1+k)^2}(yM_k(y)) \\ &= e^{i\frac{\pi}{2}(m-1)^2} e^{i\frac{\pi}{2}k^2}(yM_k(e^{i\pi(m-1)}y)) \\ &= e^{i\frac{\pi}{2}(m-1)^2} y e^{i\frac{\pi}{2}k^2}(M_k(e^{i\pi(m-1)}y)). \end{aligned}$$

Collecting everything, we have

Theorem 4.2. *The kernel $t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y^2} e^{-i\langle x,y\rangle}$ in the Laplace domain is*

$$\begin{aligned} \mathcal{L}(t^{m/2-1} e^{i\frac{\pi}{2}\Gamma_y^2} e^{-it\langle x,y\rangle}) &= \frac{\Gamma(m/2)}{4\sqrt{+}}\Big((1+i)U_m^1 + (1-i)U_m^2 \\ &\qquad + e^{i\frac{\pi}{2}(m-1)^2}((1+i)U_m^3 + (1-i)U_m^4)\Big), \end{aligned}$$

with

$$U_m^1 = \frac{s + \sqrt{+} - iyx}{(s + i\langle x,y\rangle)^{m/2}}; \qquad U_m^2 = \frac{s + \sqrt{+} + iyx}{(s - i\langle x,y\rangle)^{m/2}};$$

$$U_m^3 = \frac{(-1)^{m-1}(\sqrt{+} - s) + iyx}{(s + (-1)^{m-1} i\langle x,y\rangle)^{m/2}}; \qquad U_m^4 = \frac{(-1)^{m-1}(s - \sqrt{+}) + iyx}{(s - (-1)^{m-1} i\langle x,y\rangle)^{m/2}},$$

where $\sqrt{+} = \sqrt{s^2 + |x|^2|y|^2}$.

When $m=2$,

$$\mathcal{L}(e^{i\frac{\pi}{2}\Gamma_y^2}e^{-it\langle x,y\rangle}) = \frac{1}{2\sqrt{+}}\left(\frac{\sqrt{+}}{s-i\langle x,y\rangle}+\frac{s-iyx}{s+i\langle x,y\rangle}\right).$$

By formula (2.3), (2.8), and the convolution formula (2.13), the kernel equals, putting $t=1$,

$$K_{2,\Gamma^2}(x,y) = e^{i\langle x,y\rangle} + J_0(|x||y|) + ix\wedge y\int_0^1 e^{-i\langle x,y\rangle(1-\tau)}J_0(|x||y|\tau)d\tau.$$

In the following, we analyze each term in Theorem 4.2 in detail. By formula (2.4), (2.13) and (2.8), letting $t=1$, we get $U_m^1, U_m^2, U_m^3, U_m^4$ in the time domain as

$$\begin{aligned}
K_{U_m^1} &= \frac{e^{-i\langle x,y\rangle}}{\Gamma(m/2)} + \frac{1}{\Gamma(m/2-1)}\int_0^1 \tau^{m/2-2}e^{-i\langle x,y\rangle\tau}J_0(|x||y|(1-\tau))d\tau \\
&\quad +\frac{ix\wedge y}{\Gamma(m/2)}\int_0^1 e^{-i\langle x,y\rangle}J_0(|x||y|(1-\tau))d\tau, \\
K_{U_m^2} &= \frac{e^{i\langle x,y\rangle}}{\Gamma(m/2)} + \frac{1}{\Gamma(m/2-1)}\int_0^1 \tau^{m/2-2}e^{i\langle x,y\rangle\tau}J_0(|x||y|(1-\tau))d\tau \\
&\quad -\frac{ix\wedge y}{\Gamma(m/2)}\int_0^1 e^{i\langle x,y\rangle}J_0(|x||y|(1-\tau))d\tau, \\
K_{U_m^3} &= (-1)^{m-1}(\frac{1}{\Gamma(m/2)}e^{i(-1)^m\langle x,y\rangle} \\
&\quad -\frac{1}{\Gamma(m/2-1)}\int_0^1 \tau^{m/2-2}e^{i(-1)^m\langle x,y\rangle\tau}J_0(|x||y|(1-\tau))d\tau) \\
&\quad -\frac{ix\wedge y}{\Gamma(m/2)}\int_0^1 e^{i(-1)^m\langle x,y\rangle}J_0(|x||y|(1-\tau))d\tau,
\end{aligned}$$

$$\begin{aligned}
K_{U_m^4} &= (-1)^{m-1}(-\frac{1}{\Gamma(m/2)}e^{i(-1)^{m-1}\langle x,y\rangle} \\
&\quad +\frac{1}{\Gamma(m/2-1)}\int_0^1 \tau^{m/2-2}e^{i(-1)^{m-1}\langle x,y\rangle\tau}J_0(|x||y|(1-\tau))d\tau) \\
&\quad -\frac{ix\wedge y}{\Gamma(m/2)}\int_0^1 e^{i(-1)^{m-1}\langle x,y\rangle}J_0(|x||y|(1-\tau))d\tau.
\end{aligned}$$

Theorem 4.3. *Let $m\geq 2$. For $x,y\in\mathbb{R}^m$, the generalized Fourier kernel is given by*

$$K_{m,\Gamma^2}(x,y) = \frac{\Gamma(m/2)}{4}\Big((1+i)K_{U_m^1} + (1-i)K_{U_m^2}$$

$$+e^{i\frac{\pi}{2}(m-1)^2}((1+i)K_{U_m^3}+(1-i)K_{U_m^4})\Big).$$

There exists a constant c such that

$$|K_{m,\Gamma^2}(x,y)| \leq c(1+|x||y|).$$

Proof. This follows from the fact that $J_0(y)$ and $e^{i\langle x,y\rangle}$ are bounded functions and $|x\wedge y| \leq |x||y|$. $\square$

4.3 Closed expression for $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i\langle x,y\rangle}$

In this section, we consider the more general case. We act with $G(\Gamma_y)$ on the Fourier kernel. Here $G(x)$ is a polynomial with integer coefficients,

$$G(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_k \in \mathbb{Z}.$$

Using the fact that $e^{i\frac{\pi}{2}j}$ is 4-periodic in j,

$$e^{i\frac{\pi}{2}G(\Gamma_y)}M_k(y) = e^{i\frac{\pi}{2}G(-k)}M_k(y)$$

and

$$G(4n+k) \equiv G(k)(\mathrm{mod} 4),$$

we have

$$\begin{aligned} e^{i\frac{\pi}{2}G(\Gamma_y)}f(y) &= e^{i\frac{\pi}{2}G(0)}f_0 + e^{i\frac{\pi}{2}G(-1)}f_1 + e^{i\frac{\pi}{2}G(-2)}f_2 + e^{i\frac{\pi}{2}G(-3)}f_3 \\ &= i^{G(0)}f_0 + i^{G(-1)}f_1 + i^{G(-2)}f_2 + i^{G(-3)}f_3, \end{aligned}$$

with each f_k defined in (4.4). By

$$e^{i\frac{\pi}{2}G(\Gamma_y)}(yM_k(y)) = e^{i\frac{\pi}{2}G(m-1+k)}(yM_k)$$

and

$$G(4n+k+m-1) \equiv G(k+m-1)(\mathrm{mod} 4),$$

we have

$$\begin{aligned} & e^{i\frac{\pi}{2}G(\Gamma_y)}g(y) \\ = {} & \frac{iy}{s+\sqrt{+}}\left(e^{i\frac{\pi}{2}G(m-1)}f_0 + e^{i\frac{\pi}{2}G(m)}f_1 + e^{i\frac{\pi}{2}G(m+1)}f_2 + e^{i\frac{\pi}{2}G(m+2)}f_3\right)x \\ = {} & \frac{iy}{s+\sqrt{+}}\left(i^{G(m-1)}f_0 + i^{G(m)}f_1 + i^{G(m+1)}f_2 + i^{G(m+2)}f_3\right)x. \end{aligned}$$

Collecting everything and applying (4.6), we get

Theorem 4.4. *For $G(x) \in \mathbb{Z}[x]$, the Laplace transform of*

$$t^{m/2-1}e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-it\langle x,y\rangle}$$

is given by

$$\mathcal{L}(t^{m/2-1}e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-it\langle x,y\rangle}) = \frac{\Gamma(m/2)}{8}\left(A_m^1 B C_m^T + \frac{iy}{s+\sqrt{+}} A_m^2 B C_m^T x\right)$$

with A_m^1, A_m^2, B, C_m the matrices given by

$$\begin{aligned}
A_m^1 &= \begin{pmatrix} i^{G(0)} & i^{G(-1)} & i^{G(-2)} & i^{G(-3)} \end{pmatrix},\\
A_m^2 &= \begin{pmatrix} i^{G(m-1)} & i^{G(m)} & i^{G(m+1)} & i^{G(m+2)} \end{pmatrix},\\
B &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix},\\
C_m &= \begin{pmatrix} f(y) & f_\alpha(y) & f_\beta(y) & f_\gamma(y) \end{pmatrix}.
\end{aligned}$$

Remark 4.4. We can get the regular Fourier kernel $e^{-i\langle x,y\rangle}$ by setting $G(x) = 0$ or $4x$ for dimension $m \geq 2$. When $G = 2x^2$, we get the inverse Fourier kernel $e^{i\langle x,y\rangle}$ for even dimension. When $G(x) = \pm x$, it is the Clifford-Fourier transform studied in Chapter 3 and [29].

Remark 4.5. In general, the Clifford-Fourier transform here will not satisfy the similar properties as the classical Fourier transform for the partial derivatives. In fact, the property

$$\partial_{x_i} K(x,y) = y_i K(x,y), \qquad i = 1, \cdots, m.$$

is strict that it uniquely determines the kernel $K(x,y)$ up to a constant. In our case, when $G(x) = x$ and $m = 2$, the kernel

$$K_2(x,y) = e^{-i\frac{\pi}{2}\Gamma_y}e^{-i\langle x,y\rangle},$$

satisfies

$$\begin{aligned}
\partial_{x_1} K(x,y) &= y_2 e_1 e_2 K(x,y),\\
\partial_{x_2} K(x,y) &= -y_1 e_1 e_2 K(x,y).
\end{aligned}$$

As the constant term of the polynomial will only contribute a constant factor to the integral kernel, in the following we only consider polynomials without constant term

$$G(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x, \qquad a_k \in \mathbb{Z}.$$

By

$$G(4n+k) \equiv G(k)(\text{mod}4),$$

it reduces to four cases $G(k)(\text{mod}4)$, $k = 0, 1, 2, 3$. The set $\{x^m\} \cup \{1\}, m \in \mathbb{N}$ is a basis for polynomials over the ring of integers. We consider the four cases on this basis

$$\begin{aligned} x^j &= 0, \qquad \text{when} \quad x = 0; \\ x^j &= 1, \qquad \text{when} \quad x = 1; \\ x^j &\equiv \begin{cases} 2(\text{mod}4), & \text{when } j = 1 \text{ and } x = 2; \\ 0(\text{mod}4), & \text{when } j \geq 2 \text{ and } x = 2; \end{cases} \\ x^j &\equiv \begin{cases} 1(\text{mod}4), & \text{when } j \text{ is even and } x = 3; \\ 3(\text{mod}4), & \text{when } j \text{ is odd and } x = 3. \end{cases} \end{aligned}$$

For each $G(x)$, we denote $\frac{G(1)+G(-1)}{2} = s_0 = \sum_{j=0}^{\lfloor n/2 \rfloor} a_{2j}$ and $\frac{G(1)-G(-1)}{2} = s_1 = \sum_{j=0}^{\lfloor n/2 \rfloor} a_{2j+1}$ with n the degree of $G(x)$. We have

$$\begin{aligned} G(0) &= 0, \\ G(1) &= s_0 + s_1, \\ G(2) &\equiv 2a_1(\text{mod}4), \\ G(3) &\equiv G(-1) \equiv s_0 - s_1(\text{mod}4). \end{aligned}$$

Therefore

$$\begin{aligned} &i^{G(0)} = 1, \qquad i^{G(-1)} = i^{G(3)} = i^{s_0+3s_1}, \\ &i^{G(-2)} = i^{G(2)} = (-1)^{a_1}, \qquad i^{G(-3)} = i^{G(1)} = i^{s_0+s_1}. \end{aligned}$$

The class of integral transforms with polynomially bounded kernel is of great interest. For example, new uncertainty principles have been given for this kind of integral transforms in [52]. As we can see in Theorem 4.4, the generalized Fourier kernel is a linear combination of $f_\alpha, f_\beta, f_\gamma, f, g_\alpha, g_\beta, g_\gamma, g$. At present, very few of $f_\alpha, f_\gamma, g_\alpha, g_\gamma$ are known explicitly. The integral representations of $f_\alpha, f_\gamma, g_\alpha, g_\gamma$ are obtained in [20] but without the bound. Only in even dimensions, special linear combinations of $f_\alpha, f_\gamma, g_\alpha, g_\gamma$ are known to be polynomially bounded which is exactly the Clifford-Fourier kernel [29].

We have showed in Theorem 4.3 that f, f_β, g, g_β with polynomial bounds behave better than $f_\alpha, f_\gamma, g_\alpha, g_\gamma$. So it is interesting to consider the generalized Fourier transform whose kernel only consists of f, f_β, g, g_β. It also provides ways to define hypercomplex Fourier transforms with polynomially bounded kernel in odd dimensions. We will hence characterize polynomials such that $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i\langle x,y\rangle}$ are only linear combination of f, f_β, g, g_β.

For fixed m, the kernel is a linear sum of f, f_β, g, g_β when the polynomial $G(x) \in \mathbb{Z}[x]$ satisfies the following conditions

$$\begin{cases} i^{G(0)} - ii^{G(-1)} - i^{G(-2)} + ii^{G(-3)} = 0, \\ i^{G(0)} + ii^{G(-1)} - i^{G(-2)} - ii^{G(-3)} = 0, \\ i^{G(m-1)} - ii^{G(m)} - i^{G(m+1)} + ii^{G(m+2)} = 0, \\ i^{G(m-1)} + ii^{G(m)} - ii^{G(m+1)} - ii^{G(m+2)} = 0. \end{cases} \tag{4.7}$$

We find that (4.7) is equivalent with

$$\begin{cases} G(0) \equiv G(-2)(\text{mod}4), \\ G(-1) \equiv G(-3)(\text{mod}4), \\ G(m-1) \equiv G(m+1)(\text{mod}4), \\ G(m) \equiv G(m+2)(\text{mod}4). \end{cases} \tag{4.8}$$

As $G(k)(\text{mod}4)$ is uniquely determined by $G(0), G(-1), G(-2)$ and $G(-3)$, the first two formulas in (4.8) imply the last two formulas for all $m \geq 2$ automatically. Now (4.8) becomes

$$\begin{cases} i^{G(0)} = 1 = i^{G(-2)} = (-1)^{a_1}, \\ i^{G(-1)} = i^{s_0+3s_1} = i^{G(-3)} = i^{s_0+s_1}. \end{cases}$$

It follows that the kernel only consists of f, f_β, g, g_β if and only if a_1 and s_1 are even. We have the following

Theorem 4.5. *Let $m \geq 2$. For $x, y \in \mathbb{R}^m$ and a polynomial $G(x)$ with integer coefficients, the kernel $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i\langle x,y\rangle}$ is a linear combination of f, f_β, g, g_β in the Laplace domain if and only if a_1 and $\frac{G(1)-G(-1)}{2}$ are even. Furthermore, the generalized Fourier kernel is bounded and equals*

$$\frac{1+i^{G(1)}}{2}e^{-i\langle x,y\rangle} + \frac{1-i^{G(1)}}{2}K^\pi(x,y),$$

with $K^{\pi}(x,y)$ the fractional Clifford-Fourier kernel in Chapter 3. When $m \geq 2$ is even, the kernel is

$$\frac{1+i^{G(1)}}{2}e^{-i\langle x,y\rangle} + \frac{1-i^{G(1)}}{2}e^{i\langle x,y\rangle}.$$

When $m \geq 2$ is odd, there exists a constant c which is independent of m such that

$$|e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i\langle x,y\rangle}| \leq c(1+|x||y|). \tag{4.9}$$

Proof. We only need to prove that the generalized Fourier kernel is

$$\frac{1+i^{s_0+s_1}}{2}e^{-i\langle x,y\rangle} + \frac{1-i^{s_0+s_1}}{2}K^{\pi}(x,y).$$

In fact, by verification, we have,

$$(e^{i0})^{m-1}A_m^1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = A_m^2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}; \qquad (e^{i\pi})^{m-1}A_m^1\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = A_m^2\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},$$

and

$$A_m^1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = 2+2i^{s_0+s_1}; \qquad A_m^1\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = 2-2i^{s_0+s_1}.$$

By Remark 4.3, $f+(e^{i0})^{m-1}g$ is the kernel K_0 and $f_\beta+(e^{i\pi})^{m-1}g_\beta$ is the fractional Clifford-Fourier kernel K^{π}. The bound (4.9) follows from the integral expression of f, f_β, g, g_β in the time domain. □

Remark 4.6. The case $G(x) = x^2$ is a special case of this theorem.

In the following, we consider the generalized Fourier kernel which has polynomial bound and consists of $f_\alpha, f_\beta, f_\gamma, f, g_\alpha, g_\beta, g_\gamma, g$. For even dimension, we already know the Clifford-Fourier kernel has a polynomial bound. If the polynomial $G(x)$ satisfies

$$(-i)^{m-1}A_m^1\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix} = A_m^2\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}; \quad i^{m-1}A_m^1\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix} = A_m^2\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix},$$

(4.10)

by Remark 4.3, $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i\langle x,y\rangle}$ is a linear combination of the Clifford-Fourier kernel and some function bounded by $c(1+|x||y|)$. Hence it has a polynomial bound as well. When $m=4j$, (4.10) becomes

$$i(1-i^{s_0+3s_1+1}-(-1)^{a_1}+i^{s_0+s_1+1}) = i^{s_0+3s_1}+i-i^{s_0+s_1}-i(-1)^{a_1}$$

and

$$-i(1+i^{s_0+3s_1+1}-(-1)^{a_1}-i^{s_0+s_1+1}) = i^{s_0+3s_1}-i-i^{s_0+s_1}+i(-1)^{a_1}.$$

It shows that (4.10) is true for any $G(x)\in\mathbb{Z}[x]$ when $m=4j$. When $m=4j+2$, (4.10) becomes

$$-i(1-i^{s_0+3s_1+1}-(-1)^{a_1}+i^{s_0+s_1+1}) = i^{s_0+s_1}+i(-1)^{a_1}-i^{s_0+3s_1}-i$$

and

$$i(1+i^{s_0+3s_1+1}-(-1)^{a_1}-i^{s_0+s_1+1}) = i^{s_0+s_1}-i(-1)^{a_1}-i^{s_0+3s_1}+i.$$

It also shows that (4.10) is true for any $G(x)\in\mathbb{Z}[x]$ when $m=4j+2$. Now we have

Theorem 4.6. *Let $m\geq 2$ be even. For $x,y\in\mathbb{R}^m$ and any polynomial $G(x)$ with integer coefficients, the kernel $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i\langle x,y\rangle}$ has a polynomial bound, i.e. there exists a constant c which is independent of $G(x)$ such that*

$$|e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i\langle x,y\rangle}|\leq c(1+|x||y|)^{\frac{m-2}{2}}.$$

At the end of this section, we give the formal generating function of the even dimensional generalized Fourier kernels for a class of polynomials. We define

$$H(x,y,a,G)=\sum_{m=2,4,6,\cdots}\frac{K_{m,G}(x,y)a^{m/2-1}}{\Gamma(m/2)}.$$

Theorem 4.7. *Let $m\geq 2$ be even. For $x,y\in\mathbb{R}^m$ and any polynomial $G(x)$ with integer coefficients, the formal generating function of the even dimensional generalized Fourier kernel is given by*

$$H(x,y,a,G)$$

$$
\begin{aligned}
&= \frac{1 - i^{G(-1)+1} - (-1)^{G'(0)} + i^{G(1)+1}}{2} \Bigg(\cos(\sqrt{|x|^2|y|^2 - (\langle x,y\rangle + a)^2}) \\
&\quad -(x \wedge y - a) \frac{\sin\sqrt{|x|^2|y|^2 - (\langle x,y\rangle + a)^2}}{\sqrt{|x|^2|y|^2 - (\langle x,y\rangle + a)^2}} \Bigg) \\
&\quad + \frac{1 + i^{G(-1)+1} - (-1)^{G'(0)} - i^{G(1)+1}}{2} \Bigg(\cos(\sqrt{|x|^2|y|^2 - (\langle x,y\rangle - a)^2}) \\
&\quad +(x \wedge y + a) \frac{\sin\sqrt{|x|^2|y|^2 - (\langle x,y\rangle - a)^2}}{\sqrt{|x|^2|y|^2 - (\langle x,y\rangle - a)^2}} \Bigg) \\
&\quad + \frac{1 + i^{G(-1)} + (-1)^{G'(0)} + i^{G(1)}}{2} e^{-(i\langle x,y\rangle - a)} \\
&\quad + \frac{1 - i^{G(-1)} + (-1)^{G'(0)} - i^{G(1)}}{2} e^{i\langle x,y\rangle + a}.
\end{aligned}
$$

Proof. When m is even, the generalized Fourier kernel is

$$
\begin{aligned}
& e^{i\frac{\pi}{2}G(\Gamma_y)} e^{-i\langle x,y\rangle} \\
&= \frac{1}{2}\Bigg((1 - i^{s_0+3s_1+1} - (-1)^{a_1} + i^{s_0+s_1+1})(f_\alpha + e^{i\frac{-\pi}{2}(m-1)} g_\gamma) \\
&\quad +(1 + i^{s_0+3s_1+1} - (-1)^{a_1} - i^{s_0+s_1+1})(f_\gamma + e^{i\frac{\pi}{2}(m-1)} g_\alpha) \\
&\quad +(1 + i^{s_0+3s_1} + (-1)^{a_1} + i^{s_0+s_1}) e^{-i\langle x,y\rangle} + (1 - i^{s_0+3s_1} \\
&\quad +(-1)^{a_1} - i^{s_0+s_1}) e^{i\langle x,y\rangle} \Bigg),
\end{aligned}
$$

with $s_0 = \sum_{j=0}^{\lfloor n/2 \rfloor} a_{2j}$ and $s_1 = \sum_{j=0}^{\lfloor n/2 \rfloor} a_{2j+1}$.

By $s_0 + 3s_1 \equiv s_0 - s_1 \equiv G(-1) (\mathrm{mod} 4)$, $s_0 + s_1 = G(1)$, $a_1 = G'(0)$ and because $\frac{\Gamma(m/2)}{2}(f_\alpha + e^{i\frac{-\pi}{2}(m-1)} g_\gamma)$ and $\frac{\Gamma(m/2)}{2}(f_\gamma + e^{i\frac{\pi}{2}(m-1)} g_\alpha)$ are the Clifford-Fourier kernel $K^{\frac{-\pi}{2}}$ and $K^{\frac{\pi}{2}}$ in the Laplace domain, the result follows from the generating function of Clifford-Fourier kernel, see Chapter 3, Theorem 3.6. ☐

Remark 4.7. When $G(x) = x$, we get the generating function of the Clifford-Fourier kernel.

For the case that the coefficients of $G(x)$ are not integers but fractions, we write $G_1(x) = cG(x)$ in which c is the least common multiple of each denominator of $G(x)$. So $G_1(x)$ is a polynomial with integer coefficients. We only need to compute $e^{i\frac{\pi}{2c}G_1(\Gamma_y)} f(y)$ and $e^{i\frac{\pi}{2c}G_1(\Gamma_y)} g(y)$. The same method will also work but f and g split into $4c$ parts.

Where there is matter, there is geometry.

Johannes Kepler

5
Clifford-Fourier transform on hyperbolic space

In Chapter 3 and Chapter 4, the radial Laplace transform technique has led naturally to the solution of the action of $e^{i\frac{\pi}{2}\Gamma}$ and related exponentials of polynomial expressions in Γ, where Γ is the angular Dirac operator. Our technique had the merit of reusing the well-known Poisson and Szegö kernel results for the unit ball in the Laplace domain, connecting these results with the a priori very different world of exponential waves in which Fourier transforms and their generalisations live.

When we considered generalisations of the Fourier theory to non-Euclidean geometries of constant curvature, Lobachevsky and spherical, we were surprised to discover how these correspond, merely through some renaming of variables, to the Laplace domain results of the flat, Euclidean case; the purpose of the present chapter is therefore to establish these similarities and to state and prove new results on generalised Fourier transforms that follow from them.

This chapter is organized as follows. In section 5.1, we introduce the hyperboloid model, the Poincaré model and the Helgason-Fourier transform on each model. In section 5.2, we define the generalized Fourier transform. In section 5.3, we use the result of the Clifford-Fourier kernel in the Laplace domain to give the explicit expressions of the generalized kernels of even dimension and determine the generating function of even dimension. In section 5.4, we point out that parallel results can be obtained for the unit ball model.

5.1 Hyperbolic space and Fourier transform

There are several models of hyperbolic geometry including the Klein, Poincaré, hyperboloid, upper-half space and hemisphere models. In this section, we introduce the hyperboloid model and the Helgason-Fourier transform on it, the Poincaré model and the generalized Helgason-Fourier transform associated to the variants of Laplace-Beltrami operator.

5.1.1 Hyperboloid model and Fourier transform

The hyperbolic space here is given by the upper sheet of a two-sheeted hyperboloid embedded in the Minkowski space $\mathbb{R}^{1,m}$. The Minkowski space is a $m+1$-dimensional pseudo-Riemannian manifold equipped with a nondegenerate bilinear form

$$[x, y] = x_0y_0 - x_1y_1 - \ldots - x_my_m.$$

We consider the upper part of the pseudo-sphere of radius 1

$$H^m = \{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{1,m}, [x, x] = 1, x_0 > 0\}.$$

This space is invariant under $SO(1, m)$, the Lorentz group in $m + 1$-dimensions. The isometry group $SO(1, m)$ acts transitively on H^m and $SO(m)$ fixes the origin $(1, 0, \ldots, 0)$. The hyperbolic space H^m can be identified with the rank one symmetric space $SO_e(1, m)/SO(m)$ or $Spin_e(1, m)/Spin(m)$. Here $Spin_e(1, m), Spin(m)$ are the two-fold coverings of $SO(1, m)$ and $SO(m)$. The hyperbolic space H^m is contractible and admits a unique spin structure as it is a Riemannian symmetric space of noncompact type.

Hyperbolic space can also be given by the hyperbolic parametrization:

$$H^m = \{\Omega = (t, p) \in \mathbb{R}^{1,m}, (t, p) = (\cosh r, \omega \sinh r), r \geq 0, \omega \in S^{m-1}\}.$$

Now one has

$$dt = \sinh r dr, \qquad dp = \omega \cosh r dr + \sinh r d\omega$$

and the metric induced on H^m by the Lorenzian metric on $\mathbb{R}^{1,m}$

$$dl^2 = -dt^2 + dp^2,$$

is

$$ds^2 = dr^2 + \sinh^2 r dw^2$$

where dw^2 is the metric on the sphere S^{m-1} [3]. It is also known that the geodesic sphere in H^m is an Euclidean sphere. The Laplace-Beltrami operator on the hyperbolic space H^m is

$$\Delta_{H^m} = \partial_r^2 + (m-1)\frac{\cosh r}{\sinh r}\partial_r + \frac{1}{\sinh^2 r}\Delta_{S^{m-1}}.$$

For $\theta \in S^{m-1}$ and λ a real number, the Fourier transform on H^m is defined as [3]

$$\hat{f}(\lambda, \theta) = \int_{H^m} h_{\lambda,\theta}(\Omega) f(\Omega) d\Omega = \int_{H^m} h_{\lambda,\theta}(\Omega) f(\Omega) \sinh^{m-1} r dr d\omega.$$

Here

$$h_{\lambda,\theta} = [\Omega, \wedge(\theta)]^{i\lambda - \frac{m-1}{2}} = (\cosh r - \sinh r\langle\omega, \theta\rangle)^{i\lambda - \frac{m-1}{2}}$$

are the eigenfunctions of the Laplacian-Beltrami operator with

$$-\Delta_{H^m} h_{\lambda,\theta} = (\lambda^2 + \frac{(m-1)^2}{4}) h_{\lambda,\theta},$$

$\wedge(\theta)$ is the point $(1, \theta) \in \mathbb{R}^{1,m}$ and $\langle\omega, \theta\rangle$ is the usual Euclidean inner product. The Fourier inversion is given by

$$f(\Omega) = \int_{-\infty}^{\infty} \int_{S^{m-1}} \overline{h_{\lambda,\theta}(\Omega)} \hat{f}(\lambda, \theta) \frac{d\theta d\lambda}{|c(\lambda)|^2},$$

where $c(\lambda)$ is the Harish-Chandra function

$$\frac{1}{|c(\lambda)|^2} = \frac{1}{2(2\pi)^m} \frac{|\Gamma(i\lambda + \frac{m-1}{2})|^2}{|\Gamma(i\lambda)|^2}.$$

For more details on hyperbolic space, see [55] and [56].

5.1.2 Poincaré model and Helgason transform

In this subsection, we introduce another model, i.e. the Poincaré model. Let H^m be the unit ball B^m of $\mathbb{R}^m$, with usual Poincaré metric

$$ds^2 = \frac{4(dx_1^2 + \ldots + dx_m^2)}{(1-|x|^2)^2}.$$

A family of differential operators indexed by $v \in \mathbb{R}$

$$\Delta_v := \frac{1-|x|^2}{4}\{(1-|x|^2)\sum_{j=1}^m \frac{\partial^2}{\partial x_j^2} - 2v\sum_{j=1}^m x_j\frac{\partial}{\partial x_j} + v(2-m-v)\}$$

are the variants of the Laplace-Beltrami operator on the hyperbolic space which is recovered for $v = 2-m$. Note that Δ_v is a self-adjoint operator in $L^2(B^m, d\mu_v)$ with

$$d\mu_v(x) = 2^{2-\mu}(1-|x|^2)^{v-2}dx.$$

The generalized Helgason-Fourier transform associated to Δ_v is developed in a parallel way to the Fourier transform on the hyperbolic space. For a function $f \in C_0^\infty(B^m)$, it is defined by

$$\hat{f}(\lambda,\zeta) = \int_{B^m} f(x)e_{-\lambda,\zeta}(x)d\mu_v(x),$$

with

$$e_{\lambda,\zeta}(x) = \frac{(1-|x|^2)^{(1-v+i\lambda)/2}}{|x-\zeta|^{m-1+i\lambda}}, \qquad x \in B^m$$

for $\lambda \in \mathbb{C}$ and $\zeta \in S^{m-1}$. For more details, we refer to [69].

5.2 Generalized Fourier transform

In this chapter, the homogeneous vector bundle over the hyperbolic space $Spin_e(1,m)/Spin(m)$ is constructed by the finite dimensional representation $\mathcal{C}\ell_{0,m}^c$ of $Spin(m)$. For more details about homogeneous vector bundles over a symmetric space we refer to [95]. For simplicity, the space could be considered as $L^2(H^m) \otimes \mathcal{C}\ell_{0,m}$.

Now for the homogeneous vector bundle, like the Euclidean case, we use the angular Dirac operator to generalize the Fourier transform. We define

$$\mathcal{F}(f)(\lambda,\theta) = \int_{H^m} e^{ip\Gamma_\theta} h_{\lambda,\theta}(\Omega) f(\Omega) d\Omega$$

with $p \in \mathbb{R}$. The operator $e^{ip\Gamma}$ is a unitary operator semigroup on $L^2(H^m) \otimes \mathcal{C}\ell_{0,m}$, because Γ is a self-adjoint operator on the Euclidean sphere and the sphere on the hyperbolic space is the Euclidean sphere. It follows that the new transform is unitary, too. The new kernel

$$K_m^p(\omega,\theta,r,\lambda) = e^{ip\Gamma_\theta} h_{\lambda,\theta}$$

is still an eigenfunction of the hyperbolic Laplacian,

$$-\Delta_{H^m} e^{ip\Gamma_\theta} h_{\lambda,\theta} = (\lambda^2 + \frac{(m-1)^2}{4}) e^{ip\Gamma_\theta} h_{\lambda,\theta}.$$

The Plancherel measure is still the same since the Gamma operator commutes with radial functions. Equally, in the vein of [28], we can define

$$\mathcal{F}(f)(\lambda,\theta) = \int_{H^m} e^{i\frac{\pi}{2}P(\Gamma_\theta)} h_{\lambda,\theta}(\Omega) f(\Omega) d\Omega$$

with $P(x)$ a polynomial. We can choose $P(x)$ such that $P(\Gamma)$ is the Casimir operator of the $Spin(m)$ representation. When $P(\Gamma) = \Delta_{S^{m-1}}$, we have defined new Fourier transforms for the scalar case.

Remark 5.1. A different Fourier transform for vector fields and spinors on hyperbolic space has also been studied in [2], [19]. Their methods are similar to the scalar case, based on the G-invariant differential operator, the radial system of vector fields and the spherical Fourier transform. The spherical Fourier transform finally reduces to the Jacobi transform [66].

5.3 The generalized Fourier kernel on the hyperboloid

Let $\omega, \theta \in S^{m-1}$, $s = \frac{\cosh r}{\sinh r}$ and $k = \frac{1}{2} + i\lambda$. The generalized Fourier kernel is

$$\begin{aligned} K_m^p(\omega,\theta,r,\lambda) &= e^{ip\Gamma_\theta}[\cosh r - \sinh r \langle \omega,\theta\rangle]^{i\lambda - \frac{m-1}{2}} \\ &= (\sinh r)^{i\lambda - \frac{m-1}{2}} e^{ip\Gamma_\theta} (s - \langle\omega,\theta\rangle)^{i\lambda - \frac{m-1}{2}}, \end{aligned}$$

here $\langle\omega,\theta\rangle$ is the usual Euclidean inner product. In order to get the closed expression of the generalized kernel, we introduce an operator T, which is the $(k-1)$-th order Riemann-Liouville fractional integral of f with reference point at infinity up to a constant factor [63],

$$(Tf)(s) = \int_s^{+\infty} (u-s)^{k-1} f(u)du.$$

We express the kernel using the beta function and T as

$$\begin{aligned}
& \frac{1}{(s-\langle\omega,\theta\rangle)^{m/2-k}} \\
= \ & \frac{1}{B(\frac{m}{2}-k,k)} T((u-\langle\omega,\theta\rangle)^{m/2}) \\
= \ & \frac{1}{B(\frac{m}{2}-k,k)} \int_s^{+\infty} \frac{(u-s)^{k-1}}{(u-\langle\omega,\theta\rangle)^{m/2}} du. \qquad (5.1)
\end{aligned}$$

Indeed, we have

$$\begin{aligned}
& \int_s^{+\infty} \frac{(u-s)^{k-1}}{(u-\langle\omega,\theta\rangle)^{m/2}} du \\
= \ & \int_0^{+\infty} \frac{v^{k-1}}{(v+s-\langle\omega,\theta\rangle)^{m/2}} dv \\
= \ & \int_0^{+\infty} \frac{(w(s-\langle\omega,\theta\rangle))^{k-1}}{((w(s-\langle\omega,\theta\rangle))+s-\langle\omega,\theta\rangle)^{m/2}} d(w(s-\langle\omega,\theta\rangle)) \\
= \ & \int_0^{+\infty} \frac{w^{k-1}}{(w+1)^{m/2}} (s-\langle\omega,\theta\rangle)^{k-m/2} dw,
\end{aligned}$$

where we have used $s \geq 1$ in the second step. Changing variables by $w = \frac{\theta}{1-\theta}$, $w+1 = \frac{1}{1-\theta}$, we obtain

$$\int_0^{+\infty} \frac{w^{k-1}}{(w+1)^{m/2}} dw = \int_0^1 \frac{\theta^{k-1}/(1-\theta)^{k-1}}{(1-\theta)^{-m/2}(1-\theta)^2} d\theta$$

which is the beta function $B(\frac{m}{2}-k,k)$. Note that $B(a,b)$ converges when $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$ which means formula (5.1) requires $m \geq 2$.

5.3.1 The case $m = 2$

By Theorem 3.1, we have

$$e^{ip\Gamma_\theta} \frac{1}{(u-\langle\omega,\theta\rangle)^{m/2}}$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{-}}\left(\frac{u+\sqrt{-}+e^{-ip}\theta\omega}{(e^{-ip}(u\cos p+i\sqrt{-}\sin p-\langle\omega,\theta\rangle))^{m/2}}\right. \\
&\qquad \left. -e^{imp}\frac{u-\sqrt{-}+e^{-ip}\theta\omega}{(e^{ip}(u\cos p-i\sqrt{-}\sin p-\langle\omega,\theta\rangle))^{m/2}}\right) \qquad (5.2)
\end{aligned}
$$

with $\sqrt{-}=\sqrt{u^2-|\omega|^2|\theta|^2}$, which is the Clifford-Fourier kernel in the Laplace domain in Theorem 3.1. In our case, $|\omega|=|\theta|=1$. When $m=2$, formula (5.2) reduces to

$$
\begin{aligned}
e^{ip\Gamma_\theta}\frac{1}{u-\langle\omega,\theta\rangle} &= \frac{1}{2\sqrt{-}}\left(\frac{u+\sqrt{-}+e^{-ip}\theta\omega}{e^{-ip}(u\cos p+i\sqrt{-}\sin p-\langle\omega,\theta\rangle)}\right. \\
&\qquad \left. -\frac{u-\sqrt{-}+e^{-ip}\theta\omega}{e^{-ip}(u\cos p-i\sqrt{-}\sin p-\langle\omega,\theta\rangle)}\right) \\
&= \frac{ue^{-ip}-\langle\omega,\theta\rangle-ie^{-ip}\theta\omega\sin p}{e^{-ip}[(u-\langle\omega,\theta\rangle\cos p)^2+(\omega\wedge\theta)^2\sin^2 p]} \\
&= \frac{1}{u-\langle\omega,\theta\rangle\cos p+i(\omega\wedge\theta)\sin p}.
\end{aligned}
$$

Furthermore, when $p=\frac{\pi}{2}$, it becomes

$$
e^{i\frac{\pi}{2}\Gamma_\theta}\frac{1}{u-\langle\omega,\theta\rangle} = \frac{u-i\theta\wedge\omega}{u^2-|\omega\wedge\theta|^2}.
$$

By partial fraction decomposition and (5.1), we get the generalized kernel in dimension 2 when $p=\frac{\pi}{2}$

$$
\begin{aligned}
K_2^{\frac{\pi}{2}}(\omega,\theta,r,\lambda) &= \frac{(\sinh r)^{i\lambda-\frac{1}{2}}}{B(1-k,k)}T\left(e^{i\frac{\pi}{2}\Gamma_\theta}\frac{1}{u-\langle\omega,\theta\rangle}\right) \\
&= \frac{(\sinh r)^{i\lambda-\frac{1}{2}}}{2}\left(\left(1-\frac{i\theta\wedge\omega}{|\omega\wedge\theta|}\right)\frac{1}{(s-|\omega\wedge\theta|)^{\frac{1}{2}-i\lambda}}\right. \\
&\qquad \left.+\left(1+\frac{i\theta\wedge\omega}{|\omega\wedge\theta|}\right)\frac{1}{(s+|\omega\wedge\theta|)^{\frac{1}{2}-i\lambda}}\right). \qquad (5.3)
\end{aligned}
$$

Similarly, we can get the generalized kernel for general p.

5.3.2 The case m even

In this section, we give the explicit expression of $K_m^{\frac{\pi}{2}}(\omega,\theta,r,\lambda)$ when m is even. Let $N_1=\sqrt{-}+i\langle\theta,\omega\rangle$, $N_2=\sqrt{-}-i\langle\theta,\omega\rangle$. When $p=\frac{\pi}{2}$, formula (5.2) reduces to

$$\begin{aligned}
& e^{i\frac{\pi}{2}\Gamma_\theta}\frac{1}{(u-\langle\omega,\theta\rangle)^{m/2}} \\
= \; & \frac{1}{2\sqrt{-}}\left(\frac{u+\sqrt{-}-i\theta\omega}{(\sqrt{-}+i\langle\omega,\theta\rangle)^{m/2}} - i^m\frac{u-\sqrt{-}-i\theta\omega}{(\sqrt{-}-i\langle\omega,\theta\rangle)^{m/2}}\right) \\
= \; & A_m + B_m - i\theta\wedge\omega C_m,
\end{aligned}$$

with

$$\begin{aligned}
A_m &= \frac{1}{2\sqrt{-}}\left(\frac{1}{N_1^{\frac{m}{2}-1}} + i^m\frac{1}{N_2^{\frac{m}{2}-1}}\right); \\
B_m &= \frac{1}{2\sqrt{-}}\left(\frac{u}{N_1^{\frac{m}{2}}} - \frac{u}{N_2^{\frac{m}{2}}}\right) = uA_{m+2}; \\
C_m &= \frac{1}{2\sqrt{-}}\left(\frac{1}{N_1^{\frac{m}{2}}} - i^m\frac{1}{N_2^{\frac{m}{2}}}\right) = A_{m+2}.
\end{aligned}$$

and $\sqrt{-} = \sqrt{u^2-|\omega|^2|\theta|^2}$. So it is sufficient to study A_m. When m is even, each A_m is a rational function in u. When $m=4q$, we have

$$\begin{aligned}
A_m &= \frac{1}{2\sqrt{-}}\left(\frac{1}{N_1^{\frac{m}{2}-1}} + i^m\frac{1}{N_2^{\frac{m}{2}-1}}\right) \\
&= \frac{1}{2\sqrt{-}}(N_2^{2q-1}+N_1^{2q-1})\cdot\frac{1}{(u^2-|\omega\wedge\theta|^2)^{\frac{m}{2}-1}}.
\end{aligned}$$

The first factor of A_m is a polynomial in u of degree $2q-2$. Indeed, we have

$$\begin{aligned}
& \frac{1}{2\sqrt{-}}\left(N_1^{2q-1}+N_2^{2q-1}\right) \\
= \; & \frac{1}{2\sqrt{-}}\left(\sum_{j=0}^{2q-1}\binom{2q-1}{j}(\sqrt{-})^j(i\langle\omega,\theta\rangle)^{2q-1-j}(1+(-1)^{2q-1-j})\right) \\
= \; & \frac{1}{2\sqrt{-}}\left(2\sum_{j=0}^{q-1}\binom{2q-1}{2j+1}(\sqrt{-})^{2j+1}(i\langle\omega,\theta\rangle)^{2q-1-2j-1}\right) \\
= \; & \sum_{j=0}^{q-1}\binom{2q-1}{2j+1}(u^2-|\omega|^2|\theta|^2)^j(i\langle\omega,\theta\rangle)^{2q-2j-2}.
\end{aligned}$$

Now for $m = 4q$, we have

$$\begin{aligned}
& A_m \\
= & \sum_{j=0}^{q-1} \binom{2q-1}{2j+1} \frac{(u^2 - |\omega|^2|\theta|^2)^j (i\langle\omega,\theta)^{2q-2j-2}}{(u^2 - |\omega\wedge\theta|^2)^{2q-1}} \\
= & \sum_{j=0}^{q-1}\sum_{l=0}^{j} \binom{2q-1}{2j+1}\binom{j}{l}(-1)^{q-1-l} \frac{(u^2 - |\omega\wedge\theta|^2)^l (\langle\omega,\theta\rangle)^{2q-2j-2+2j-2l}}{(u^2 - |\omega\wedge\theta|^2)^{2q-1}} \\
= & \sum_{j=0}^{q-1}\sum_{l=0}^{j} \binom{2q-1}{2j+1}\binom{j}{l}(-1)^{q-l-1} \frac{\langle\omega,\theta\rangle^{2q-2l-2}}{(u^2 - |\omega\wedge\theta|^2)^{2q-1-l}} \\
= & \sum_{l=0}^{q-1}\left(\sum_{j=l}^{q-1} \binom{2q-1}{2j+1}\binom{j}{l}(-1)^{q-l-1}\right) \frac{\langle\omega,\theta\rangle^{2q-2l-2}}{(u^2 - |\omega\wedge\theta|^2)^{2q-l-1}}
\end{aligned}$$

where we have used $|\omega|^2|\theta|^2 = |\omega\wedge\theta|^2 + \langle\omega,\theta\rangle^2$ in the second equality. Furthermore, we have

$$\sum_{j=l}^{q-1} \binom{2q-1}{2j+1}\binom{j}{l}(-1)^{q-l-1} = (-1)^{q-l-1} \frac{2^{2q-2l-2}\Gamma(2q-l-1)}{\Gamma(l+1)\Gamma(2q-2l-1)}$$

which yields

$$A_m = \sum_{l=0}^{q-1} (-1)^{q-l-1} \frac{2^{2q-2l-2}\Gamma(2q-l-1)}{\Gamma(l+1)\Gamma(2q-2l-1)} \frac{\langle\omega,\theta\rangle^{2q-2l-2}}{(u^2 - |\omega\wedge\theta|^2)^{2q-l-1}}.$$

When $m = 4q+2$, A_m can be computed similarly. Using $B_m = uA_{m+2}$ and $C_m = A_{m+2}$, we have, for m even,

$$\begin{aligned}
A_m &= \sum_{l=0}^{\lfloor \frac{m}{4}-\frac{3}{4}\rfloor} \frac{2^{m/2-2l-2}\Gamma(m/2-l-1)}{\Gamma(l+1)\Gamma(m/2-2l-1)} \frac{(-i\langle\omega,\theta\rangle)^{m/2-2l-2}}{(u^2 - |\omega\wedge\theta|^2)^{m/2-l-1}}; \\
B_m &= \sum_{l=0}^{\lfloor \frac{m}{4}-\frac{1}{2}\rfloor} \frac{2^{m/2-2l-1}\Gamma(m/2-l)}{\Gamma(l+1)\Gamma(m/2-2l)} \frac{(-i\langle\omega,\theta\rangle)^{m/2-2l-1}u}{(u^2 - |\omega\wedge\theta|^2)^{m/2-l}}; \\
C_m &= \sum_{l=0}^{\lfloor \frac{m}{4}-\frac{1}{2}\rfloor} \frac{2^{m/2-2l-1}\Gamma(m/2-l)}{\Gamma(l+1)\Gamma(m/2-2l)} \frac{(-i\langle\omega,\theta\rangle)^{m/2-2l-1}}{(u^2 - |\omega\wedge\theta|^2)^{m/2-l}}.
\end{aligned}$$

Remark 5.2. The closed expressions of A_m, B_m and C_m correspond to Theorem 4.3 in [29] which is in the time domain.

Subsequently, by partial fraction decomposition

$$F(s)=\frac{b(s)}{\prod_{j=1}^{l}(s-a_j)^{m_j}}=\sum_{j=1}^{l}\sum_{n}^{m_j}\frac{r_{j,n}}{(s-a_j)^n}$$

where the poles a_j of $F(s)$ are distinct and the degree of b less than $\sum_{j=1}^{l} m_j$ and

$$r_{j,n}=\frac{1}{(m_j-n)!}\frac{d^{m_j-n}}{ds^{m_j-n}}\bigg(F(s)(s-a_j)^{m_j}\bigg)\bigg|_{s=a_j}$$

and (5.1), we have

$$\begin{aligned}
& G_m(s,\alpha): \\
= \;& T\bigg(\frac{1}{(u^2-\alpha^2)^m}\bigg)(s) \\
= \;& T\bigg(\sum_{l=1}^{m}\frac{(-1)^m(m)_{m-l}}{(m-l)!(2\alpha)^{2m-l}}\bigg(\frac{(-1)^l}{(u-\alpha)^l}+\frac{1}{(u+\alpha)^l}\bigg)\bigg)(s) \\
= \;& \sum_{l=1}^{m}\frac{(-1)^m(m)_{m-l}B(l-k,k)}{(m-l)!(2\alpha)^{2m-l}}\bigg(\frac{(-1)^l}{(s-\alpha)^{l-k}}+\frac{1}{(s+\alpha)^{l-k}}\bigg)
\end{aligned}$$

as well as

$$\begin{aligned}
& H_m(s,\alpha): \\
= \;& T\bigg(\frac{u}{(u^2-\alpha^2)^m}\bigg)(s) \\
= \;& T\bigg(\frac{1}{(u+\alpha)^{m-1}(u-\alpha)^m}\bigg)(s)-\alpha G_m(s,\alpha) \\
= \;& \sum_{l=1}^{m}\frac{(-1)^{m-l}(m-1)_{m-l}}{(m-l)!(2\alpha)^{2m-l-1}}\frac{B(l-k,k)}{(s-\alpha)^{l-k}} \\
& +\sum_{j=1}^{m-1}\frac{(-1)^m(m)_{m-j}}{(m-1-j)!(2\alpha)^{2m-j}}\frac{B(j-k,k)}{(s+\alpha)^{j-k}}-\alpha G_m(s,\alpha).
\end{aligned}$$

Now, we have all necessary material to give the explicit expression of the even dimensional kernels.

Theorem 5.1. *When $p=\frac{\pi}{2}$, the kernel of generalized Fourier transform on the hyperboloid in even dimension $m>2$ is given by*

$$K_m^{\frac{\pi}{2}}(\omega,\theta,r,\lambda)$$

$$= \frac{(\sinh r)^{i\lambda - \frac{m-1}{2}}}{B(\frac{m-1}{2} - i\lambda, \frac{1}{2} + i\lambda)} (A_m^*(s,\alpha) + B_m(s,\alpha)^* - i\theta \wedge \omega C_m^*(s,\alpha))$$

where

$$\begin{aligned} A_m^*(s,\alpha) &= \sum_{l=0}^{\lfloor \frac{m}{4} - \frac{3}{4} \rfloor} \frac{2^{m/2-2l-2}\Gamma(m/2-l-1)}{\Gamma(l+1)\Gamma(m/2-2l-1)} \\ &\quad \times (-i\langle \omega, \theta \rangle)^{m/2-2l-2} G_{m/2-l-1}(s,\alpha); \\ B_m^*(s,\alpha) &= \sum_{l=0}^{\lfloor \frac{m}{4} - \frac{1}{2} \rfloor} \frac{2^{m/2-2l-1}\Gamma(m/2-l)}{\Gamma(l+1)\Gamma(m/2-2l)} \\ &\quad \times (-i\langle \omega, \theta \rangle)^{m/2-2l-1} H_{m/2-l}(s,\alpha); \\ C_m^*(s,\alpha) &= \sum_{k=0}^{\lfloor \frac{m}{4} - \frac{1}{2} \rfloor} \frac{2^{m/2-2l-1}\Gamma(m/2-l)}{\Gamma(l+1)\Gamma(m/2-2l)} \\ &\quad \times (-i\langle \omega, \theta \rangle)^{m/2-2l-1} G_{m/2-l}(s,\alpha). \end{aligned}$$

$s = \frac{\cosh r}{\sinh r}$, $k = \frac{1}{2} + i\lambda$ *and* $\alpha = |\omega \wedge \theta|$.

Remark 5.3. The kernel for general p can be obtained similarly.

5.3.3 Generating function for the even dimensional kernels

Classical orthogonal polynomials can be defined by their generating functions. One can obtain the orthogonal polynomial by differentiating the generating function. In this section we compute the formal generating function for the even dimensional generalized Fourier kernel. By the formal generating function, it is possible to get the generalized Fourier kernel by differentiation. The generating function is defined by

$$G_p(\omega,\theta,r,\lambda,\epsilon) = \sum_{m=2,4,6,\cdots} B\left(\frac{m}{2} - k, k\right) \hat{K}_m^p(\omega,\theta,r,\lambda)(e^{-ip}\epsilon)^{m/2-1},$$

where $\hat{K}_m^p(\omega,\theta,r,\lambda) = \dfrac{K_m^p(\omega,\theta,r,\lambda)}{(\sinh r)^{i\lambda - \frac{m-1}{2}}}$. Formally, the generating function of the Clifford-Fourier kernel in the Laplace domain is

$$\sum_{m=2,4,6,\cdots} \frac{(e^{-ip}\epsilon)^{m/2-1}}{2\sqrt{-}} \left(\frac{u + \sqrt{-} + e^{-ip}\theta\omega}{(e^{-ip}(u\cos p + i\sqrt{-}\sin p - \langle \omega, \theta \rangle))^{m/2}} \right.$$

$$
\begin{aligned}
& -e^{imp}\frac{u-\sqrt{-}+e^{-ip}\theta\omega}{(e^{ip}(u\cos p-i\sqrt{-}\sin p-\langle\omega,\theta\rangle))^{m/2}}\Bigg) \\
= \quad & \frac{1}{2\sqrt{-}}\Bigg(\frac{u+\sqrt{-}+e^{-ip}\theta\omega}{e^{-ip}(u\cos p+i\sqrt{-}\sin p-\langle\omega,\theta\rangle)-(e^{-ip}\epsilon)} \\
& -\frac{u-\sqrt{-}+e^{-ip}\theta\omega}{e^{-ip}(u\cos p-i\sqrt{-}\sin p-\langle\omega,\theta\rangle))-(e^{-ip}\epsilon)}\Bigg) \\
= \quad & \frac{u-i\theta\wedge\omega\sin p-\langle\omega,\theta\rangle\cos p-\epsilon e^{ip}}{(u\cos p-\langle\omega,\theta\rangle-\epsilon)^2-(i\sqrt{-}\sin p)^2} \\
= \quad & \frac{u-i\theta\wedge\omega\sin p-\langle\omega,\theta\rangle\cos p-\epsilon e^{ip}}{(u-(\langle\omega,\theta\rangle+\epsilon)\cos p)^2-[-(\langle\omega,\theta\rangle+\epsilon)^2+|\omega|^2|\theta|^2]\sin^2 p}
\end{aligned}
$$

with $\sqrt{-}=\sqrt{u^2-1}$. The formal generating function of the generalized Fourier kernel is then

$$
\begin{aligned}
& G_p(\omega,\theta,r,\lambda,\epsilon) \\
= \quad & T\Bigg(\frac{u-i\theta\wedge\omega\sin p-\langle\omega,\theta\rangle\cos p-\epsilon e^{ip}}{(u-(\langle\omega,\theta\rangle+\epsilon)\cos p)^2-[-(\langle\omega,\theta\rangle+\epsilon)^2+|\omega|^2|\theta|^2]\sin^2 p}\Bigg).
\end{aligned}
$$

By partial fraction decomposition and (5.1), we have

Theorem 5.2. *The formal generating function for the generalized Fourier kernel of even dimension $m \geq 2$ is given by*

$$
G_p(\omega,\theta,r,\lambda,\epsilon)=\frac{\pi\sec(i\lambda\pi)}{2\tilde{c}}\Bigg(\frac{\tilde{b}+\tilde{c}-\tilde{a}}{(s-\tilde{c}-\tilde{b})^{\frac{1}{2}-i\lambda}}+\frac{\tilde{a}-\tilde{b}+\tilde{c}}{(s-\tilde{b}+\tilde{c})^{\frac{1}{2}-i\lambda}}\Bigg)
$$

where

$$
\begin{aligned}
\tilde{a} &= i\theta\wedge\omega\sin p+\langle\omega,\theta\rangle\cos p+\epsilon e^{ip}, \\
\tilde{b} &= (\langle\omega,\theta\rangle+\epsilon)\cos p, \\
\tilde{c} &= \sqrt{|\omega|^2|\theta|^2-(\langle\omega,\theta\rangle+\epsilon)^2}\sin p.
\end{aligned}
$$

5.3.4 Generalized Fourier kernel associated to $P(\Gamma)$

In this section, we consider a more general case, acting with $P(\Gamma)$ on the Fourier kernel. Here $P(x)$ is a polynomial with integer coefficients:

$$
P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x, \qquad a_k\in\mathbb{Z}.
$$

Now, the generalized Fourier kernel is given by

$$\begin{aligned} & e^{i\frac{\pi}{2}P(\Gamma_\theta)}[\cosh r - \sinh r\langle\omega,\theta\rangle]^{i\lambda-\frac{m-1}{2}} \\ = \ & (\sinh r)^{i\lambda-\frac{m-1}{2}} e^{i\frac{\pi}{2}P(\Gamma_\theta)}(s-\langle\omega,\theta\rangle)^{i\lambda-\frac{m-1}{2}} \\ = \ & \frac{(\sinh r)^{i\lambda-\frac{m-1}{2}}}{B(\frac{m}{2}-k,k)} T(e^{i\frac{\pi}{2}P(\Gamma_\theta)}(s-\langle\omega,\theta\rangle)^{-\frac{m}{2}}) \end{aligned}$$

By Theorem 4.4 in Chapter 4, we have

$$\begin{aligned} & e^{i\frac{\pi}{2}P(\Gamma_\theta)}\frac{1}{(u-\langle\omega,\theta\rangle)^{\frac{m}{2}}} \\ = \ & \frac{1-i^{P(-1)+1}-(-1)^{P'(0)}+i^{P(1)+1}}{2}\left(e^{i\frac{-\pi}{2}\Gamma_\theta}\frac{1}{(u-\langle\omega,\theta\rangle)^{\frac{m}{2}}}\right) \\ & +\frac{1+i^{P(-1)+1}-(-1)^{P'(0)}-i^{P(1)+1}}{2}\left(e^{i\frac{\pi}{2}\Gamma_\theta}\frac{1}{(u-\langle\omega,\theta\rangle)^{\frac{m}{2}}}\right) \\ & +\frac{1+i^{P(-1)}+(-1)^{P'(0)}+i^{P(1)}}{2}\frac{1}{(u-\langle\omega,\theta\rangle)^{\frac{m}{2}}} \\ & +\frac{1-i^{P(-1)}+(-1)^{P'(0)}-i^{P(1)}}{2}\frac{1}{(u+\langle\omega,\theta\rangle)^{\frac{m}{2}}}. \end{aligned}$$

By (5.1) and (5.3), we get the generalized kernel of dimension 2

$$\begin{aligned} & K_2^{\frac{\pi}{2}}(\omega,\theta,r,\lambda) \\ = \ & (\sinh r)^{i\lambda-\frac{1}{2}}\Bigg(\frac{1-i^{P(-1)+1}-(-1)^{P'(0)}+i^{P(1)+1}}{4} \\ & \times\Bigg(\left(1+\frac{i\theta\wedge\omega}{|\omega\wedge\theta|}\right)\frac{1}{(s-|\omega\wedge\theta|)^{\frac{1}{2}-i\lambda}}+\left(1-\frac{i\theta\wedge\omega}{|\omega\wedge\theta|}\right)\frac{1}{(s+|\omega\wedge\theta|)^{\frac{1}{2}-i\lambda}}\Bigg) \\ & +\frac{1+i^{P(-1)+1}-(-1)^{P'(0)}-i^{P(1)+1}}{4}\Bigg(\left(1-\frac{i\theta\wedge\omega}{|\omega\wedge\theta|}\right)\frac{1}{(s-|\omega\wedge\theta|)^{\frac{1}{2}-i\lambda}} \\ & +\left(1+\frac{i\theta\wedge\omega}{|\omega\wedge\theta|}\right)\frac{1}{(s+|\omega\wedge\theta|)^{\frac{1}{2}-i\lambda}}\Bigg) \\ & +\frac{1+i^{P(-1)}+(-1)^{P'(0)}+i^{P(1)}}{2}\frac{1}{(s-\langle\omega,\theta\rangle)^{\frac{1}{2}-i\lambda}} \\ & +\frac{1-i^{P(-1)}+(-1)^{P'(0)}-i^{P(1)}}{2}\frac{1}{(s+\langle\omega,\theta\rangle)^{\frac{1}{2}-i\lambda}}\Bigg). \end{aligned}$$

The generating function can be obtained similarly.

5.4 Generalized Fourier kernel on the unit ball

We define our generalized Fourier transform on the unit ball as

$$\mathcal{F}f(\lambda,\zeta) = \int_{B^m} e^{ip\Gamma_\zeta} e_{-\lambda,\zeta}(x) f(x) d\mu_v(x),$$

with $p \in \mathbb{R}$.

The generalized kernel is

$$\begin{aligned} & K_m^p(x,\zeta,\lambda) \\ = \; & e^{ip\Gamma_\zeta} e_{-\lambda,\zeta}(x) \\ = \; & e^{ip\Gamma_\zeta} \frac{(1-|x|^2)^{(1-v+i\lambda)/2}}{|x-\zeta|^{m-1+i\lambda}} \\ = \; & \frac{(1-|x|^2)^{(1-v+i\lambda)/2}}{2^{\frac{m-1+i\lambda}{2}}} e^{ip\Gamma_\zeta} \frac{1}{\left(\frac{1+|x|^2}{2} - \langle x,\zeta\rangle\right)^{\frac{m-1+i\lambda}{2}}}. \end{aligned}$$

For simplicity, we set $p = \frac{\pi}{2}$. Substituting $\frac{1+|x|^2}{2}$ for s in (5.3), we get the kernel for dimension 2:

$$\begin{aligned} & K_2^{\frac{\pi}{2}}(x,\zeta,\lambda) \\ = \; & \frac{(1-|x|^2)^{(1-v+i\lambda)/2}}{2} \left(\left(1 - \frac{i\zeta\wedge x}{|\zeta\wedge x|}\right) \frac{1}{((1+|x|^2) - 2|\zeta\wedge x|)^{\frac{1+i\lambda}{2}}} \right. \\ & \left. + \left(1 + \frac{i\zeta\wedge x}{|\zeta\wedge x|}\right) \frac{1}{((1+|x|^2) + 2|\zeta\wedge x|)^{\frac{1+i\lambda}{2}}} \right). \end{aligned}$$

Note here that the k for the hyperboloid and unit ball models are different. Substituting $\frac{1+|x|^2}{2}$ for s and $-\lambda/2$ for λ in Theorem 5.1, we get the kernel for all even dimensions:

Theorem 5.3. *When $p = \frac{\pi}{2}$, the kernel of generalized Fourier transform on the unit ball in even dimension $m > 2$ is given by*

$$\begin{aligned} K_m^{\frac{\pi}{2}}(x,\zeta,\lambda) \; = \; & \frac{(1-|x|^2)^{(1-v+i\lambda)/2}}{2^{\frac{m-1+i\lambda}{2}} B(\frac{m-1-i\lambda}{2}, \frac{1-i\lambda}{2})} \\ & \times \left(A_m^*(z,\beta) + B_m^*(z,\beta) - i\zeta\wedge x C_m^*(z,\beta) \right) \end{aligned}$$

where $k = \frac{1-i\lambda}{2}$, $z = \frac{1+|x|^2}{2}$, $\beta = |\zeta\wedge x|$ and A_m^*, B_m^*, C_m^* defined in Theorem 5.1.

The generating function of the even dimensional kernel can also be obtained by substituting $\frac{1+|x|^2}{2}$ for s in Theorem 5.2 as was similarly done in Theorem 5.3.

It is true that Fourier had the opinion that the principal end of mathematics was public utility and the explanation of natural phenomena; but a philosopher as he is should have known that the unique end of science is the honor of the human mind and that from this point of view a question of [the theory of] number is as important as a question of the system of the world.

Karl Jacobi

6 Radially deformed Fourier kernel and Dunk dihedral kernel

This chapter focusses on two generalized Fourier transforms in particular, namely the Dunkl transform [30,42] and the (κ, a)-generalized Fourier transform [5]. Both transforms depend on a number of parameters, and as such it is an open problem, except for certain special cases, to find concrete formulas for their integral kernels.

Our aim in this chapter is to develop the Laplace method for finding explicit expressions as well as integral expressions for these kernels. Explicit expressions can be obtained when some of the arising parameters take on rational or integer values. The integral expressions we will obtain are valid in full generality and are expressed in terms of the generalized Mittag-Leffler function (see [73] or the subsequent Definition 6.1).

Essentially our method works as follows. Consider the following series expansion, for $x, y \in \mathbb{R}^m$

$$K^m(x,y) = 2^\lambda \Gamma(\lambda+1) \sum_{j=0}^{\infty} (-i)^j \frac{\lambda+j}{\lambda} z^{-\lambda} J_{j+\lambda}(z) C_j^\lambda(\xi)$$

with $\lambda = (m-2)/2$, $z = |x||y|$, $\xi = \langle x, y\rangle/z$, $J_{j+\lambda}(z)$ the Bessel function and $C_j^\lambda(\xi)$ the Gegenbauer polynomial. It is not so easy to recognize that this is the classical Fourier kernel $e^{-i\langle x,y\rangle}$.

However, when we introduce an auxiliary variable t in the kernel as follows

$$K^m(x,y,t) = 2^\lambda \Gamma(\lambda+1) \sum_{j=0}^{\infty} (-i)^j \frac{\lambda+j}{\lambda} z^{-\lambda} J_{j+\lambda}(tz) C_j^\lambda(\xi)$$

we can take the Laplace transform in t of $K^m(x,y,t)$. Simplifying the result by making use of the Poisson kernel (see subsequent Theorem 6.2) then yields

$$\mathcal{L}(K^m(x,y,t)) = \Gamma(\lambda+1) \frac{1}{(s+i\langle x,y\rangle)^{\lambda+1}}.$$

of which we immediately compute the inverse Laplace transform as

$$K^m(x,y,t) = t^{\frac{m-2}{2}} e^{-it\langle x,y\rangle}$$

and the classical Fourier kernel is recovered by putting $t=1$.

We develop this method in full generality for the Dunkl kernel related to dihedral groups, as well as for the (κ,a)- generalized Fourier transform when $\kappa=0$. The restriction to dihedral groups is necessary, because only then the Poisson kernel for the Dunkl Laplace operator is known, see [44] or subsequent Theorem 6.11.

Let us describe our main results. The Laplace transform of the $(0,a)$-generalized Fourier transform is obtained in Theorem 6.4. When $a=2/n$ and m is even, the result is a rational function and we can apply partial fraction decomposition to obtain an explicit expression, see Theorem 6.6. We prove that the kernel for $a=2/n$ is bounded by 1 in Theorem 6.9, for both even and odd dimensions. For arbitrary a, the integral expression in terms of the generalized Mittag-Leffler function is given in Theorem 6.10.

The Laplace transform of the Dunkl kernel for dihedral groups is obtained in Theorem 6.12. Two alternative integral expressions for the Dunkl kernel, again in terms of the generalized Mittag-Leffler function, are given in Theorem 6.13 and 6.14.

This chapter is organized as follows. After the necessary preliminaries in section 6.1, we first study the (κ,a)-generalized Fourier transform for $\kappa=0$ in section 6.2. In section 6.3 we then study the Dunkl kernel for dihedral groups. We also show how our methods can be applied to the Dunkl Bessel function.

6.1 Dunkl operator and generalized Fourier transform

In this section, we give a brief overview of the theory of Dunkl operators and the (κ, a)-generalized Fourier transform. Most of these results are taken from [44], [80] and [5]. We use the notation $\langle \cdot, \cdot \rangle$ for the standard inner product on $\mathbb{R}^m$ and $|\cdot|$ for the associated norm. For a non-zero vector $\alpha \in \mathbb{R}^m$, the reflection r_α in the hyperplane orthogonal to α is defined by

$$r_\alpha(x) = x - 2\frac{\langle \alpha, x \rangle}{|\alpha|^2}\alpha.$$

A reduced root system $\mathcal{R}$ is a finite set of non-zero vectors in $\mathbb{R}^m$ such that $r_\alpha \mathcal{R} = \mathcal{R}$ and $\mathbb{R}\alpha \cap \mathcal{R} = \{\pm\alpha\}$ for all $\alpha \in \mathcal{R}$. The finite reflection group generated by $\{r_\alpha : \alpha \in \mathcal{R}\}$ is a subgroup of the orthogonal group $O(m)$ which is called a Coxeter group. Standard root systems are A_{n-1}, B_n and the root system associated to the dihedral groups. We give the latter as an example which will be used later.

Example 6.1. In the Euclidean space $\mathbb{R}^2$, let $d \in O(2, \mathbb{R})$ be the rotation over $2\pi/k$ and e the reflection at the y-axis. The group I_k generated by d and e consists of all orthogonal transformations which preserve a regular k-sided polygon centered at the origin. The group I_k is a finite reflection group which is usually called dihedral group.

We define the action of G on functions by

$$(g \cdot f)(x) := f(g^{-1} \cdot x), \qquad x \in \mathbb{R}^m, g \in G.$$

A multiplicity function $\kappa : \mathcal{R} \to \mathbb{C}$ is a function invariant under the action of G. Furthermore, set $\mathcal{R}_+ := \{\alpha \in \mathcal{R} : \langle \alpha, \beta \rangle > 0\}$ for some $\beta \in \mathbb{R}^m$ such that $\langle \alpha, \beta \rangle \neq 0$ for all $\alpha \in \mathcal{R}$. From now on, fix the positive subsystem $\mathcal{R}_+$ and the multiplicity function κ. The Dunkl operator T_i associated to $\mathcal{R}_+$ and κ is then defined by

$$T_i f(x) = \frac{\partial f}{\partial x_i} + \sum_{\alpha \in \mathcal{R}_+} \kappa(\alpha)\alpha_i \frac{f(x) - f(r_\alpha(x))}{\langle \alpha, x \rangle}, \qquad f \in C^1(\mathbb{R}^m)$$

where α_i is the i-th coordinate of α. All the T_i commute with each other. They reduce to the corresponding partial derivatives when

$\kappa = 0$. The Dunkl Laplacian Δ_κ is then defined as $\Delta_\kappa = \sum_{i=1}^m T_i^2$. The weight function associated with the root system $\mathcal{R}$ and the multiplicity function κ is given by

$$\upsilon_\kappa(x) := \prod_{\alpha \in \mathcal{R}_+} |\langle x, \alpha\rangle|^{2\kappa(\alpha)}.$$

It is G-invariant and homogeneous of degree $2\langle\kappa\rangle$ where the index $\langle\kappa\rangle$ of the multiplicity function κ is defined as

$$\langle\kappa\rangle := \sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha = \frac{1}{2}\sum_{\alpha\in\mathcal{R}} \kappa_\alpha.$$

We also denote by $\mathcal{H}_j(\upsilon_\kappa)$ the space of Dunkl harmonics of degree j, i.e. $\mathcal{H}_j(\upsilon_\kappa) = \mathcal{P}_j \cap \ker\Delta_\kappa$ with $\mathcal{P}_j$ the space of homogeneous polynomials of degree j. There exists a unique linear and homogeneous isomorphism on polynomials which intertwines the algebra of Dunkl operators and the algebra of usual partial differential operators, i.e. $V_\kappa(\mathcal{P}_j) = \mathcal{P}_j, \quad V_\kappa|_{\mathcal{P}_0} = id$ and $T_\xi V_\kappa = V_\kappa \partial_\xi$ for all $\xi \in \mathbb{R}^m$. In the following, we denote by $P_j(G; x, y)$ the reproducing kernel of $\mathcal{H}_j(\upsilon_\kappa)$ and by $P(G; x, y)$ the Poisson kernel. For $j \in \mathbb{N}$ and $|y| \leq |x| = 1$, we have [44]

$$P_j(G; x, y) = \frac{j + \lambda_\kappa}{\lambda_\kappa} V_\kappa[C_j^{\lambda_\kappa}(\langle\cdot, \frac{y}{|y|}\rangle)](x)|y|^j, \tag{6.1}$$

and

$$\begin{aligned} P(G; x, y) &= \sum_{j=0}^\infty P_j(G; x, y) = \sum_{j=0}^\infty P_j(G; x, \frac{y}{|y|})|y|^j \\ &= V_\kappa\left(\frac{1 - |y|^2}{(1 - 2\langle\cdot, y\rangle + |y|^2)^{\lambda_\kappa + 1}}\right)(x) \end{aligned} \tag{6.2}$$

where $\lambda_\kappa = \langle\kappa\rangle + \frac{m-2}{2}$. Rösler [81] proved there exists a unique positive probability-measure $\mu_x(\xi)$ on $\mathbb{R}^m$ such that

$$V_\kappa f(x) = \int_{\mathbb{R}^m} f(\xi) d\mu_x(\xi).$$

In [5], Dunkl's intertwining operator V_κ was extended to $C(B)$ with B the closed unit ball in $\mathbb{R}^m$. Denoting

$$(\tilde{V_\kappa} h) := (V_\kappa h_y)(x) = \int_{\mathbb{R}^m} h(\langle\xi, y\rangle) d\mu_x(\xi),$$

this operator satisfies

$$||\tilde{V}_\kappa h||_{L^\infty(B\times B)} \leq ||h||_{L^\infty([-1,1])}. \tag{6.3}$$

It is known that the operators T_j have a joint eigenfunction $E_\kappa(x,y)$ satisfying

$$T_j E_\kappa(x,y) = -iy_j E_\kappa(x,y), \qquad j = 1,\dots,m.$$

The function $E_\kappa(x,y)$ is called the Dunkl kernel, which is the exponential function $e^{-i\langle x,y\rangle}$ when $\kappa = 0$. This kernel together with the weight function is used to define the so-called Dunkl transform [30]$\mathcal{F}_\kappa : L^1(\mathbb{R}^m, \upsilon_\kappa) \to C(\mathbb{R}^m)$ by

$$\mathcal{F}_\kappa f(y) := c_\kappa \int_{\mathbb{R}^m} f(x)E(x,y)\upsilon_\kappa(x)dx \quad (y \in \mathbb{R}^m)$$

with $c_\kappa^{-1} = \int_{\mathbb{R}^m} e^{-|x|^2/2}\upsilon_\kappa(x)dx$ the Mehta constant associated to G. Again, when $\kappa = 0$, we recover the classical Fourier transform. The Dunkl transform shares many properties with the Fourier transform. As we have introduced in Section 2.4, see also [61], using the harmonic oscillator $-(\Delta - |x|^2)/2$, Howe found the spectral description of the Fourier transform and its eigenfunctions forming the basis of $L^2(\mathbb{R}^m)$:

$$\mathcal{F} = e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta - |x|^2)}$$

with Δ the Laplace operator. Similarly, the Dunkl transform also has the exponential notation

$$\mathcal{F}_\kappa = e^{\frac{i\pi\mu}{4}} e^{\frac{i\pi}{4}(\Delta_\kappa - |x|^2)}$$

where $\mu = m + 2\langle\kappa\rangle$, see [4]. In [5], the authors defined a radially deformed Dunkl-type harmonic oscillator

$$\Delta_{\kappa,a} = |x|^{2-a}\Delta_\kappa - |x|^a, \qquad a > 0.$$

Then the (κ, a)-generalized Fourier transform is defined by

$$\mathcal{F}_{\kappa,a} = e^{\frac{i\pi}{2a}(m-2+2\langle\kappa\rangle+a)} e^{\frac{i\pi}{2a}\Delta_{\kappa,a}}$$

in $L^2(\mathbb{R}^m, |x|^{a-2}\upsilon_\kappa(x))$. We write the (κ, a)-generalized Fourier transform as an integral transform

$$\mathcal{F}_{\kappa,a} f(y) = c_{\kappa,a} \int_{\mathbb{R}^m} B_{\kappa,a}(x,y) f(x)|x|^{a-2}\upsilon_\kappa(x)dx$$

where $c_{\kappa,\alpha}^{-1} = \int_{\mathbb{R}^m} e^{-|x|^a/a}|x|^{a-2}\upsilon_\kappa(x)dx$. The series expression of $B_{\kappa,a}(x,y)$ is given in [5] as follows,

Theorem 6.1. *For $x, y \in \mathbb{R}^m$ and $a > 0$, we have*

$$\begin{aligned} & B_{\kappa,a}(x,y) \\ = \quad & a^{\frac{2\langle\kappa\rangle+m-2}{a}}\Gamma\left(\frac{2\langle\kappa\rangle+m+a-2}{a}\right)\sum_{j=0}^{\infty} B_{\kappa,a}^{(j)}(z)P_j(G;\omega,\eta) \end{aligned}$$

where $x = |x|\omega$, $y = |y|\eta$, $z = |x||y|$, $\lambda_{\kappa,a,j} = \frac{2j+2\langle\kappa\rangle+m-2}{a}$,

$$B_{\kappa,a}^{(j)}(z) = e^{-i\frac{\pi}{2}\frac{j}{a}} z^{-\langle\kappa\rangle-m/2+1} J_{\lambda_{\kappa,a,j}}\left(\frac{2}{a}z^{a/2}\right),$$

and

$$P_j(G;\omega,\eta) := \left(\frac{\langle\kappa\rangle+j+\frac{m-2}{2}}{\langle\kappa\rangle+\frac{m-2}{2}}\right)V_\kappa[C_j^{\lambda_\kappa}(\langle\cdot,\eta\rangle)](\omega),$$

is the reproducing kernel of the space of spherical κ-harmonic polynomials of degree j.

This transform recovers the Dunkl transform when $a = 2$, the Fourier transform when $a = 2$ and $\kappa = 0$. The operator $\mathcal{F}_{0,1}$ is the unitary inversion operator of the Schrödinger model of the minimal representation of the group $O(m+1,2)$ [65]. The explicit expression of the Dunkl kernel $B_{\kappa,2}(x,y) = E_\kappa(x,y)$ is only known for the groups $\mathbb{Z}_2^m$, the root systems A_2, B_2 and some dihedral groups with integer muliplicity function κ, see [42], [44] and [33]. For the integral kernel $B_{\kappa,a}(x,y)$, except the already known Dunkl kernel, closed expressions have been found when $\kappa = 0$ and $a = \frac{2}{n}$ with $n \in \mathbb{N}$ in dimension 2, see [24]. For higher even dimension, an iterative procedure using derivatives is given there as well. Pitt's inequalities and uncertainty principles for the (κ, a)-generalized Fourier transform have been established in [53, 62] .

6.2 The (κ, a)-generalized Fourier kernel

6.2.1 Explicit expression of the kernel when $a = \frac{2}{n}$ and m even

In this section, we first establish the connection between the kernel of the $(0, a)$-generalized Fourier kernel and the Poisson kernel for the unit ball by introducing an auxiliary variable in the kernel and using the Laplace transform. Then we give the explicit formula for the kernel when $a = \frac{2}{n}$ and m even.

The kernel $K_a^m(x,y) = B_{0,a}(x,y)$ for $a > 0$ is given in Theorem 6.1 as (see also [24], [5])

$$\begin{aligned} & K_a^m(x,y) \\ = \;& a^{2\lambda/a}\Gamma\left(\frac{2\lambda+a}{a}\right)\sum_{j=0}^{\infty} e^{-\frac{i\pi j}{a}}\frac{\lambda+j}{\lambda} z^{-\lambda} J_{\frac{2(j+\lambda)}{a}}\left(\frac{2}{a}z^{a/2}\right) C_j^{\lambda}(\xi) \end{aligned}$$

with $\lambda = (m-2)/2$, $z = |x||y|$, $\xi = \langle x, y\rangle/z$ and $C_j^{\lambda}(\xi)$ the Gegenbauer polynomial. We introduce an auxiliary variable t in the kernel as follows

$$\begin{aligned} & K_a^m(x,y,t) \\ = \;& a^{2\lambda/a}\Gamma\left(\frac{2\lambda+a}{a}\right)\sum_{j=0}^{\infty} e^{-\frac{i\pi j}{a}}\frac{\lambda+j}{\lambda} z^{-\lambda} J_{\frac{2(j+\lambda)}{a}}\left(\frac{2}{a}z^{a/2}t\right) C_j^{\lambda}(\xi). \end{aligned} \tag{6.4}$$

Before we take the Laplace transform, we give the expansion of the Poisson kernel in terms of Gegenbauer polynomials.

Theorem 6.2. *[44] For $x, y \in \mathbb{R}^m$ and $|y| \le |x| = 1$, the Poisson kernel for the unit ball is*

$$\begin{aligned} P(x,y) &= \frac{1-|y|^2}{|x-y|^m} = \frac{1-|y|^2}{(1-2\xi|y|+|y|^2)^{m/2}} \\ &= \sum_{j=0}^{\infty}\frac{j+m/2-1}{m/2-1} C_j^{m/2-1}(\xi)|y|^j, \quad \xi = \langle x, \frac{y}{|y|}\rangle. \end{aligned}$$

Furthermore for $\lambda > 0$, we have

$$\frac{1-|y|^2}{(1-2\xi|y|+|y|^2)^{\lambda+1}} = \sum_{j=0}^{\infty}\frac{j+\lambda}{\lambda}C_j^{\lambda}(\xi)|y|^j. \tag{6.5}$$

It is still valid for $z \in \mathbb{C}$, $|z| < 1$ and $|\xi| < 1$, (see [74])

$$\frac{1-z^2}{(1-2\xi z+z^2)^{\lambda+1}} = \sum_{j=0}^{\infty}\frac{j+\lambda}{\lambda}C_j^{\lambda}(\xi)z^j. \tag{6.6}$$

To establish the validity of the analytic continuation of (6.5) to (6.6), note that the left-hand side of (6.6) is analytic in z in any disk centered at the origin of the complex plane that does not contain any zero of the denominator, hence analytic in $0 \leq |z| < 1$. The right-hand side of (6.6) will certainly converge to an analytic continuation of that of (6.5) for all z satisfying $|z| \leq |y| < 1$, hence for the whole unit disk.

By Theorem 6.2 and the formula from [48]

$$\begin{aligned} \mathcal{L}(J_\nu(bt)) &= \frac{1}{\sqrt{s^2+b^2}}\left(\frac{b}{s+\sqrt{s^2+b^2}}\right)^\nu, \\ &\quad \operatorname{Re}\nu > -1, \operatorname{Re}s > |\operatorname{Im}b|, \end{aligned} \tag{6.7}$$

we take the Laplace transform with respect to t in (6.4). With $u_R = e^{\frac{-i\pi}{a}}(\frac{2z^{a/2}}{aR})^{2/a}$, $r = \sqrt{s^2+(\frac{2}{a}z^{a/2})^2}$, $R = s+r$, $\lambda = (m-2)/2$, $z = |x||y|$ and $\xi = \langle x, y\rangle/z$, for $\operatorname{Re}s$ big enough, we obtain

$$\begin{aligned} &\mathcal{L}(K_a^m(x,y,t)) \\ =\ & 2^{2\lambda/a}\Gamma\left(\frac{2\lambda+a}{a}\right)\frac{1}{r}\left(\frac{1}{R}\right)^{2\lambda/a}\frac{1-u_R^2}{(1-2\xi u_R+u_R^2)^{\lambda+1}} \\ =\ & 2^{2\lambda/a}\Gamma\left(\frac{2\lambda+a}{a}\right)\frac{1}{r}\frac{R^{2/a}-\frac{e^{-2i\pi/a}(2/a)^{4/a}z^2}{R^{2/a}}}{(R^{2/a}-2\xi e^{-i\pi/a}(2/a)^{2/a}z+\frac{e^{-2i\pi/a}(2/a)^{4/a}z^2}{R^{2/a}})^{\lambda+1}} \\ =\ & 2^{2\lambda/a}\Gamma\left(\frac{2\lambda+a}{a}\right)\frac{1}{r} \\ &\times\frac{(s+r)^{2/a}-e^{-2i\pi/a}(r-s)^{2/a}}{((s+r)^{2/a}-2\xi e^{-i\pi/a}(2/a)^{2/a}z+e^{-2i\pi/a}(r-s)^{2/a})^{\lambda+1}}. \end{aligned} \tag{6.8}$$

The validity of transforming term by term in (6.4) is guaranteed by the following theorem.

Theorem 6.3. *[36] Let the function $F(s)$ be represented by a series of $\mathcal{L}$-transforms*

$$F(s) = \sum_{v=0}^{\infty} F_v(s), \quad F_v(s) = \mathcal{L}(f_v(t)),$$

where all integrals

$$\mathcal{L}(f_v) = \int_0^\infty e^{-st} f_v(t)dt = F_v(s), \quad (v = 0, 1, \cdots)$$

converge in a common half-plane $\mathrm{Re}\, s \geq x_0$. *Moreover, we require that the integrals*

$$\mathcal{L}(|f_v|) = \int_0^\infty e^{-st}|f_v(t)|dt = G_v, \quad (v = 0, 1, \cdots)$$

and the series

$$\sum_{v=0}^{\infty} G_v(x_0)$$

converge which implies that $\sum_{v=0}^{\infty} F_v(s)$ *converges absolutely and uniformly in* $\mathrm{Re}\, s \geq x_0$. *Then* $\sum_{v=0}^{\infty} f_v(t)$ *converges, absolutely, towards a function* $f(t)$ *for almost all* $t \geq 0$*; this f(t) is the original function of* $F(s)$*;*

$$\mathcal{L}\left(\sum_{v=0}^{\infty} f_v(t)\right) = \sum_{v=0}^{\infty} F_v(s).$$

Hence we can summarize our results as follows,

Theorem 6.4. *The kernel of the deformed Fourier transform in the Laplace domain is*

$$\begin{aligned} & \mathcal{L}(K_a^m(x,y,t)) \\ = \ & 2^{2\lambda/a}\Gamma\left(\frac{2\lambda+a}{a}\right)\frac{1}{r} \\ & \times \frac{(s+r)^{2/a} - e^{-2i\pi/a}(r-s)^{2/a}}{((s+r)^{2/a} - 2\xi e^{-i\pi/a}(2/a)^{2/a}z + e^{-2i\pi/a}(r-s)^{2/a})^{\lambda+1}} \end{aligned} \tag{6.9}$$

where $r = \sqrt{s^2 + (\frac{2}{a}z^{a/2})^2}$.

By direct computation, we have the following simpler expression when $m > 2$.

Corollary 6.1. *When* $\lambda > 0$*, the kernel of the deformed Fourier transform in the Laplace domain is*

$$\begin{aligned} & \mathcal{L}(K_a^m(x,y,t)) \\ = \ & -2^{2\lambda/a}\Gamma\left(\frac{2\lambda}{a}\right) \\ & \times \frac{d}{ds}\left(\frac{1}{((s+r)^{2/a} - 2\xi e^{-i\pi/a}(2/a)^{2/a}z + e^{-2i\pi/a}(r-s)^{2/a})^{\lambda}}\right) \end{aligned}$$

where $r = \sqrt{s^2 + (\frac{2}{a}z^{a/2})^2}$.

Let us now look at a few special cases. When $a = 1$, (6.9) reduces to

$$\mathcal{L}(K_1^m(x,y,t)) = \Gamma(2\lambda+1)\frac{s}{(s^2+2z+2\xi z)^{\lambda+1}}.$$

Using the formula in [48]

$$\mathcal{L}^{-1}(2^{\nu+1}\pi^{-1/2}\Gamma(\nu+3/2)a^{\nu}\sqrt{s^2+a^2}^{-2\nu-3}s) = t^{\nu+1}J_{\nu}(at),$$
$$\operatorname{Re}\nu > -1, \operatorname{Re}s > |\operatorname{Im}a| \quad (6.10)$$

and then setting $t = 1$ in $K_1^m(x,y,t)$, we reobtain the kernel

$$K_1^m(x,y) = \Gamma(\lambda+1/2)\tilde{J}_{\frac{m-3}{2}}(\sqrt{2(|x||y|+\langle x,y\rangle)})$$

with $\tilde{J}_{\nu}(z) = J_{\nu}(z)(z/2)^{-\nu}$, see [64].

When $a = 2$, (6.9) reduces to

$$\mathcal{L}(K_2^m(x,y,t)) = \Gamma(\lambda+1)\frac{1}{(s+i\xi z)^{\lambda+1}}.$$

By the inverse transform formula in [48]

$$\mathcal{L}\left(\frac{t^{k-1}e^{-\alpha t}}{\Gamma(k)}\right) = \frac{1}{(s+\alpha)^k} \qquad k > 0,$$

and then putting $t = 1$ in $K_2^m(x,y,t)$, we get the classical Fourier kernel

$$K_2^m(x,y) = e^{-i\langle x,y\rangle}.$$

We are interested in the case when $a = \frac{2}{n}$, because it has a close relationship with the Dunkl kernel and Dunkl Bessel function associated with dihedral groups which we will discuss in Section 4. When $a = \frac{2}{n}$, the Fourier kernel in the Laplace domain is

$$\mathcal{L}(K_{\frac{2}{n}}^m(x,y,t)) = \Gamma(n\lambda+1)\frac{Q_{n-1}(s)}{P_n(s)^{\lambda+1}}, \quad (6.11)$$

with

$$\begin{aligned} Q_{n-1}(s) &= \frac{(s+r)^n - e^{-in\pi}(r-s)^n}{2^n r}, \\ P_n(s) &= \frac{(s+r)^n - 2\xi e^{-in\pi/2}(n)^n z + e^{-in\pi}(r-s)^n}{2^n}. \end{aligned}$$

By direct computation, we have

$$\frac{d}{ds}P_n(s) = nQ_{n-1}(s), \tag{6.12}$$

and

$$\begin{aligned}\mathcal{L}(K^m_{\frac{2}{n}}(x,y,t)) &= \Gamma(n\lambda+1)\frac{\frac{d}{ds}P_n(s)}{n(P_n(s))^{\lambda+1}} \\ &= -\Gamma(n\lambda)\frac{d}{ds}\frac{1}{P_n(s)^{\lambda}},\end{aligned} \tag{6.13}$$

when $\lambda > 0$.

We can investigate both functions $Q_{n-1}(s)$ and $P_n(s)$ in more detail. This is done in the following lemma.

Lemma 6.1. *The function $P_n(s)$ is a polynomial of degree n in s with the factorization*

$$P_n(s) = \prod_{l=0}^{n-1}\left(s + inz^{1/n}\cos\left(\frac{q+2\pi l}{n}\right)\right),$$

where $q = \arccos(\xi)$, $\xi = \frac{\langle x,y\rangle}{|x||y|}$. The function $Q_{n-1}(s)$ is a polynomial of degree $n-1$ in s. When n is odd, $Q_{n-1}(s)$ has the factorization

$$Q_{n-1}(s) = \prod_{l=1}^{n-1}\left(s - inz^{1/n}\cos\left(\frac{l\pi}{n}\right)\right).$$

When n is even, $Q_{n-1}(s)$ has the factorization

$$Q_{n-1}(s) = \prod_{l=0,l\neq\frac{n}{2}}^{n-1}\left(s - inz^{1/n}\sin\left(\frac{l\pi}{n}\right)\right).$$

Proof. 1. We show that $P_n(s)$ is a polynomial of degree n in s,

$$\begin{aligned}&2^nP_n(s)\\ =\;& (s+r)^n - 2\xi e^{-in\pi/2}(n)^n z + e^{-in\pi}(r-s)^n\\ =\;& (s+r)^n + (-1)^n(r-s)^n - 2\xi e^{-in\pi/2}(n)^n z\\ =\;& \sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k\end{aligned}$$

$$
\begin{aligned}
&+(-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} s^{n-k} r^k - 2\xi e^{-in\pi/2}(n)^n z \\
= &\left(\sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (1+(-1)^k) \right) - 2\xi e^{-in\pi/2}(n)^n z \\
= &\ 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} s^{n-2k} (s^2 + (nz^{1/n})^2)^k - 2\xi e^{-in\pi/2}(n)^n z.
\end{aligned}
$$

Hence $2^n P_n(s)$ is a polynomial of degree n in s. The coefficient of s^n is $2\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = 2^n$.

2. We verify $2^n P_n(s_l) = 0$ with $s_l = -inz^{1/n}\cos(\frac{q+2\pi l}{n})$, $\quad l = 0, \cdots, n-1$. Denote $\xi = \cos(q) = \frac{e^{iq}+e^{-iq}}{2}$. When $\sin(\frac{q+2\pi l}{n}) \geq 0$, we have

$$
\begin{aligned}
& 2^n P_n(s_l) \\
= &\ (-inz^{1/n})^n \Bigg[\left(\cos\left(\frac{q+2\pi l}{n} \right) + i \sin\left(\frac{q+2\pi l}{n} \right) \right)^n - 2\xi \\
&+ \left(\cos\left(\frac{q+2\pi l}{n} \right) - i \sin\left(\frac{q+2\pi l}{n} \right) \right)^n \Bigg] \\
= &\ (-inz^{1/n})^n \left(e^{iq} - 2\left(\frac{e^{iq}+e^{-iq}}{2} \right) + e^{-iq} \right) \\
= &\ 0.
\end{aligned}
$$

Similarly, we have $2^n P_n(s_l) = 0$ when $\sin(\frac{q+2\pi l}{n}) < 0$. Hence, $s_l, l = 0, \cdots, n-1$ are all roots of $2^n P_n$ and we get the factorization

$$
P_n(s) = \prod_{l=0}^{n-1} \left(s + inz^{1/n} \cos\left(\frac{q+2\pi l}{n} \right) \right).
$$

3. For $2^n Q_{n-1}(s)$, we have

$$
\begin{aligned}
2^n Q_{n-1}(s) &= \frac{(s+r)^n - e^{-in\pi}(r-s)^n}{r} \\
&= \frac{1}{r}((s+r)^n - (-1)^n (r-s)^n) \\
&= \frac{1}{r} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (1 - (-1)^n (-1)^{n-k})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{r}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k+1}s^{n-2k-1}r^{2k+1} \\
&= 2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k+1}s^{n-2k-1}(s^2+(nz^{1/n})^2)^k.
\end{aligned}
$$

So $2^nQ_{n-1}(s)$ is a polynomial of degree $n-1$ in s.

4. When n is odd, $s_l = inz^{1/n}\cos(\frac{l\pi}{n}) = inz^{1/n}\sin(\frac{\pi}{2}+\frac{l\pi}{n})$, $l = 0,\cdots,n-1$ are n roots of $(2^nrQ_{n-1})(s) = 0$. Indeed, we have $r_l = \sqrt{s_l^2+(nz^{1/n})^2} = -nz^{1/n}\cos(\frac{\pi}{2}+\frac{l\pi}{n})$ and

$$
\begin{aligned}
2^nr_lQ_{n-1}(s_l) &= (s_l+r_l)^n - e^{-in\pi}(r_l-s_l)^n \\
&= (s_l+r_l)^n + (r_l-s_l)^n \\
&= (-nz^{1/n})^n(e^{-i\frac{\pi n}{2}-il\pi}+e^{i\frac{\pi n}{2}+il\pi}) \\
&= 0
\end{aligned}
$$

because n is odd. Note that $r_l = 0$ if and only if when $l = 0$. So s_l, $l = 1,\cdots,n-1$ are the $n-1$ roots of the polynomial $Q_{n-1}(s)$. Hence, we have

$$
Q_{n-1}(s) = \prod_{l=1}^{n-1}\left(s - inz^{1/n}\sin\left(\frac{\pi}{2}+\frac{l\pi}{n}\right)\right).
$$

When n is even, we verify $2^nr_lQ_{n-1}(s_l) = 0$ with $s_l = inz^{1/n}\sin(\frac{l\pi}{n})$, $l = 0,\cdots,n-1$. For $l \le \frac{n}{2}$,

$$
\begin{aligned}
2^nr_lQ_{n-1}(s_l) &= (s_l+r_l)^n - e^{-in\pi}(r_l-s_l)^n \\
&= (s_l+r_l)^n - (r_l-s_l)^n \\
&= (nz^{1/n})^n(e^{il\pi}-e^{-il\pi}) \\
&= 0.
\end{aligned}
$$

Similarly, for $l > \frac{n}{2}$, we have $2^nr_lQ_{n-1}(s_l) = 0$. Moreover, we have $r_l = 0$ if and only if $l = \frac{n}{2}$. So s_l, $l \neq \frac{n}{2}$ are the $n-1$ roots of the polynomial $Q_{n-1}(s)$. Hence we have

$$
Q_{n-1}(s) = \prod_{l=0, l\neq\frac{n}{2}}^{n-1}\left(s - inz^{1/n}\sin\left(\frac{l\pi}{n}\right)\right).
$$

$\square$

We now have all the tools necessary to compute the inverse Laplace transform. First we treat the case of dimension 2.

Theorem 6.5. *For $a = \frac{2}{n}, n \in \mathbb{N}$ and $m = 2$, we have*

$$K^2_{\frac{2}{n}}(x,y) = \frac{1}{n}\sum_{l=0}^{n-1} e^{-inz^{1/n}\cos(\frac{q+2\pi l}{n})}.$$

Proof. We have, using (6.11) and (6.12)

$$\mathcal{L}(K^2_{\frac{2}{n}}(x,y,t)) = \frac{Q_{n-1}(s)}{P_n(s)} = \frac{1}{n}\frac{\frac{d}{ds}P_n(s)}{P_n(s)} = \frac{1}{n}\sum_{l=0}^{n-1}\frac{1}{s+inz^{1/n}\cos(\frac{q+2\pi l}{n})}.$$

Taking the inverse Laplace transform and putting $t = 1$ yields the result. □

Remark 6.1. This result was previously obtained in [24] in a different way, using series multisection.

When the dimension $m > 2$, we first use (6.13) to obtain

$$K^m_{\frac{2}{n}}(x,y,t) = -\Gamma(n\lambda)\mathcal{L}^{-1}\left(\frac{d}{ds}\frac{1}{P_n(s)^\lambda}\right). \tag{6.14}$$

The inverse Laplace transform can be computed using the property of the Laplace transform

$$\mathcal{L}^{-1}\left(-\frac{d}{ds}\mathcal{L}(f(t))\right) = tf(t)$$

and the partial fraction decomposition

$$\mathcal{L}^{-1}\left(\frac{1}{P_n(s)^\lambda}\right) = \sum_{k=1}^{n}\sum_{l=1}^{\lambda}\frac{\Phi_{kl}(a_k)}{(\lambda-l)!(l-1)!}t^{\lambda-l}e^{a_k t} \tag{6.15}$$

with $a_k = -inz^{1/n}\cos(\frac{q+2\pi k}{n})$, $q = \arccos(\xi)$ and

$$\Phi_{kl}(s) = \frac{d^{l-1}}{ds^{l-1}}\left[\left(\frac{s-a_k}{P_n(s)}\right)^\lambda\right].$$

Putting $t = 1$ in (6.14) and (6.15), then yields

Theorem 6.6. *When $a = \frac{2}{n}, n \in \mathbb{N}$, the kernel of the $(0, a)$-generalized Fourier transform in even dimension $m > 2$ is given by*

$$K^m_{\frac{2}{n}}(x, y) = \Gamma(n\lambda) \sum_{k=1}^{n} \sum_{l=1}^{\lambda} \frac{\Phi_{kl}(-inz^{1/n} \cos(\frac{q+2\pi k}{n}))}{(\lambda - l)!(l-1)!} e^{-inz^{1/n} \cos(\frac{q+2\pi k}{n})}.$$

As we have given the factored form of $P_n(s)$ in Lemma 6.1, it is possible to give an explicit formula of $\Phi_{kl}(s)$ by the following result from [17].

Theorem 6.7. *Suppose $\phi(s)$ is a proper rational function having m zeros $-\sigma_h$ of multiplicity M_h and n poles $-s_k$ of multiplicity N_k,*

$$\phi(s) = \frac{\prod_{h=1}^{m}(s + \sigma_h)^{M_h}}{\prod_{k=1}^{n}(s + s_k)^{N_k}}.$$

Define the functions

$$f_k(s) = \phi(s)(s + s_k)^{N_k} = \frac{\prod_{h=1}^{m}(s + \sigma_h)^{M_h}}{\prod_{\substack{k'=1, \\ k' \neq k}}^{n}(s + s_{k'})^{N_{k'}}}, \qquad k = 1, 2, \cdots, n,$$

obtained from $\phi(s)$ by removing the factor $(s+s_k)^{N_k}$. The first derivative of $f_k(s)$ is given by

$$f_k^{(1)}(s) = f_k(s) g_k(s)$$

with

$$g_k(s) = \sum_{h=1}^{m} \frac{M_h}{s + \sigma_h} - \sum_{\substack{k'=1, \\ k' \neq k}}^{n} \frac{N_{k'}}{s + s_{k'}}.$$

The r-th derivative of g_k is given by

$$g_k^{(r)}(s) = (-1)^r r! \Bigg[\sum_{h=1}^{m} \frac{M_h}{(s + \sigma_h)^{r+1}} - \sum_{\substack{k'=1, \\ k' \neq k}}^{n} \frac{N_{k'}}{(s + s_{k'})^{r+1}} \Bigg].$$

The i-th derivative of $f_k(s)$ can be expressed by

$$f_k^{(i)} = (-1)^{i-1} f_k^{(0)} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & g_k^{(0)} \\ g_k^{(0)} & -1 & 0 & \cdots & 0 & 0 & g_k^{(1)} \\ 2g_k^{(1)} & g_k^{(0)} & -1 & \cdots & 0 & 0 & g_k^{(2)} \\ & & & \cdots & & & \\ (i-1)g_k^{(i-2)} & \binom{i-1}{2} g_k^{(i-3)} & \binom{i-1}{3} g_k^{(i-4)} & \cdots & (i-1)g_k^{(1)} & g_k^{(0)} & g_k^{(i-1)} \end{vmatrix}.$$

6.2.2 Generating function when $a = \frac{2}{n}$ and m even

For fixed $a = \frac{2}{n}$ and $n \in \mathbb{N}$, we define the formal generating function of the $(0, a)$-generalized Fourier kernel of even dimension by

$$G_{\frac{2}{n}}(x, y, \varepsilon) = \sum_{\lambda=0}^{\infty} \frac{1}{2^{n\lambda}\Gamma(n\lambda + 1)}(-2e^{-in\pi/2}(n)^n z\varepsilon)^\lambda K^m_{\frac{2}{n}}(x, y).$$

We introduce an auxiliary variable t in the generating function as

$$G_{\frac{2}{n}}(x, y, \varepsilon, t) = \sum_{\lambda=0}^{\infty} \frac{1}{2^{n\lambda}\Gamma(n\lambda + 1)}(-2e^{-in\pi/2}(n)^n z\varepsilon)^\lambda K^m_{\frac{2}{n}}(x, y, t).$$

Then we compute the Laplace transform of $G_{\frac{2}{n}}(x, y, \varepsilon, t)$, and get

$$\begin{aligned}
& \mathcal{L}(G_{\frac{2}{n}}(x, y, \varepsilon, t)) \\
= & \sum_{\lambda=0}^{\infty} \frac{1}{r} \frac{((s + r)^n - e^{-in\pi}(r - s)^n)(-2e^{-in\pi/2}(n)^n z\varepsilon)^\lambda}{((s + r)^n - 2\xi e^{-in\pi/2}(n)^n z + e^{-in\pi}(r - s)^n)^{\lambda+1}} \\
= & \frac{1}{r} \frac{(s + r)^n - e^{-in\pi}(r - s)^n}{(s + r)^n - 2(\xi + \varepsilon)e^{-in\pi/2}(n)^n z + e^{-in\pi}(r - s)^n}.
\end{aligned}$$

Comparing with Theorem 6.5, we find the only difference is that ξ in the latter becomes $\xi + \varepsilon$. Now we can give the generating function by taking the inverse Laplace transform and setting $t = 1$.

Theorem 6.8. *Let $a = 2/n$, with $n \in \mathbb{N}$. Then the formal generating function of the $(0, a)$-generalized Fourier kernel of even dimension is*

$$\begin{aligned}
G_{\frac{2}{n}}(x, y, \varepsilon) &= \sum_{\lambda=0}^{\infty} \frac{1}{2^{n\lambda}\Gamma(n\lambda + 1)}(-2e^{-in\pi}(n)^n z\varepsilon)^\lambda K^m_{\frac{2}{n}}(x, y) \\
&= \frac{1}{n} \sum_{l=0}^{n-1} e^{-inz^{1/n}\cos(\frac{\tilde{q}+2\pi l}{n})},
\end{aligned}$$

with $\tilde{q} = \arccos(\xi + \varepsilon)$.

Remark 6.2. By taking consecutive derivatives with respect to ε, we can get an alternative expression for the even dimensional kernel $K^m_{\frac{2}{n}}(x, y)$. This coincides with Proposition 2 in [24].

6.2.3 The bounds of the kernel when $a = \frac{2}{n}$ and $m \geq 2$

In this section, we prove the boundedness of the kernel $K^m_{\frac{2}{n}}(x, y)$, $m \geq 2$. This is not obvious from the explicit expansion in Theorem 6.6 as we don't know the bounds of $\Phi_{kl}(a_k)$ in (6.15). We first establish a technical lemma. Let us recall the convolution formula of the Laplace transform. Denoting $\mathcal{L}(g(t)) = G(s)$ and $\mathcal{L}(f(t)) = F(s)$, we have

$$\mathcal{L}^{-1}(G(s)F(s)) = \int_0^t g(t-\tau)f(\tau)d\tau. \qquad (6.16)$$

Lemma 6.2. *For $a_j \in \mathbb{R}, j = 1, \cdots, n$, and $k > 0$, put*

$$F_{n,k}(s) = \frac{1}{\prod_{j=1}^n (s + ia_j)^k}$$

with inverse Laplace transform

$$f_{n,k}(t) = \mathcal{L}^{-1}(F_{n,k}(s)).$$

Then

$$|f_{n,k}(t)| \leq \frac{t^{nk-1}}{\Gamma(nk)}, \qquad \forall t \in]0, \infty[.$$

Proof. We prove it by induction. By the Laplace transform formula

$$\mathcal{L}\left(\frac{t^{k-1}e^{-\alpha t}}{\Gamma(k)}\right) = \frac{1}{(s+\alpha)^k}, \qquad k > 0,$$

we have

$$f_{1,k}(t) = \frac{t^{k-1}}{\Gamma(k)} e^{-ia_1 t},$$

so

$$|f_{1,k}(t)| \leq \frac{t^{k-1}}{\Gamma(k)}.$$

When $n = 2$, by the convolution formula (6.16), we have

$$\begin{aligned} |f_{2,k}(t)| &= \left| \int_0^t \frac{(t-\tau)^{k-1} e^{-ia_1(t-\tau)}}{\Gamma(k)} f_{1,k}(\tau) d\tau \right| \\ &\leq \int_0^t \frac{(t-\tau)^{k-1}}{\Gamma(k)} |f_{1,k}(\tau)| d\tau \\ &\leq \frac{1}{\Gamma(k)^2} \int_0^t (t-\tau)^{k-1} \tau^{k-1} d\tau \end{aligned}$$

$$
\begin{aligned}
&= \frac{t^{2k-1}}{\Gamma(k)^2}\int_0^1 (1-x)^{k-1}x^{k-1}dx \\
&= \frac{t^{2k-1}}{\Gamma(2k)}
\end{aligned}
$$

where we have substituted $\tau = tx$ in the third integral. We assume

$$
|f_{n-1,k}(t)| \le \frac{t^{(n-1)k-1}}{\Gamma((n-1)k)}. \tag{6.17}
$$

Then by the convolution formula (6.16) and (6.17), we have

$$
\begin{aligned}
|f_{n,k}(t)| &\le \int_0^t \frac{(t-\tau)^{k-1}e^{-ia_n(t-\tau)}}{\Gamma(k)}|f_{n-1}(\tau)|d\tau \\
&\le \int_0^t \frac{(t-\tau)^{k-1}}{\Gamma(k)}\frac{\tau^{(n-1)k-1}}{\Gamma((n-1)k)}d\tau \\
&\le \frac{t^{nk-1}}{\Gamma(k)\Gamma((n-1)k)}\int_0^1 x^{(n-1)k-1}(1-x)^{k-1}dx \\
&\le t^{nk-1}\frac{B((n-1)k,k)}{\Gamma(k)\Gamma((n-1)k)} \\
&= \frac{t^{nk-1}}{\Gamma(nk)}.
\end{aligned}
$$

where we used the same substitution as before, and with $B(u,v)$ the beta function. □

By (6.13), when $\lambda > 0$,

$$
\begin{aligned}
&\mathcal{L}(K^m_{\frac{2}{n}}(x,y,t)) \\
&= -\Gamma(n\lambda)\frac{d}{ds}\frac{1}{(P_n(s))^\lambda} \\
&= -\Gamma(n\lambda)\frac{d}{ds}\frac{1}{\left(\prod_{l=0}^{n-1}\left(s+inz^{1/n}\cos\left(\frac{q+2\pi l}{n}\right)\right)\right)^\lambda}.
\end{aligned} \tag{6.18}
$$

Setting $t=1$, we get

$$
K^m_{\frac{2}{n}}(x,y) = \Gamma(n\lambda)f_{n,\lambda}(1)
$$

with $a_l = nz^{1/n}\cos\left(\frac{q+2\pi l}{n}\right)$ in $f_{n,\lambda}(t)$. The problem of finding an integral expression of $K^m_{\frac{2}{n}}(x,y)$ thus reduces to finding an integral expression of the function $f_{n,\lambda}(t)$.

From the Laplace transform table [48], we have

$$\mathcal{L}^{-1}\left(\frac{d}{ds}\left(\frac{1}{((s+ib)(s-ib))^{\nu+1/2}}\right)\right) = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+1/2)}t^{\nu+1}\frac{J_\nu(bt)}{b^\nu},$$
$$\operatorname{Re}\nu > -1, \operatorname{Re}s > |\operatorname{Im}b|.$$

Compared with (6.18), the Fourier kernel $K^m_{\frac{2}{n}}(x,y)$ and $f_{n,k}(t)$ could be thought of as a generalization of the Bessel function. We will see similar behavior in the Dunkl case, see Section 4.

By the inverse Laplace formula from [48],

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{\Gamma(\nu)}{(s+a)^\nu(s+b)^\nu}\right) \\ = \;& \sqrt{\pi}\left(\frac{t}{a-b}\right)^{\nu-1/2} e^{-\frac{(a+b)t}{2}} I_{\nu-1/2}\left(\frac{a-b}{2}t\right), \qquad \operatorname{Re}\nu > 0. \end{aligned}$$

we can express $f_{n,\lambda}(t)$ as the convolution of Bessel functions and exponential functions, using (6.16).

In particular, when $n = 3$, and $\operatorname{Re}s > 0$, we have

$$\begin{aligned} & f_{3,k}(t) \\ = \;& \mathcal{L}^{-1}(F_{3,k}(s)) \\ = \;& \mathcal{L}^{-1}\left(\frac{1}{\prod_{j=1}^3 (s+ia_j)^k}\right) \\ = \;& \frac{t^{3k-1}}{\Gamma(3k)} e^{ia_1 t}\Phi_2(k,k;3k; i(a_1-a_2)t, i(a_1-a_3)t) \end{aligned} \tag{6.19}$$

where $\Phi_2(c_1,c_2;c_3;w,z) = \sum_{k,l=0}^\infty \frac{(c_1)_k(c_2)_l}{(c_3)_{k+l}}\frac{w^k z^l}{k!l!}$, see [77].

Now we can give the main result of this subsection,

Theorem 6.9. *For $n \in \mathbb{N}$ and $m \geq 2$, the kernel of the $(0, 2/n)$-generalized Fourier transform satisfies*

$$|K^m_{\frac{2}{n}}(x,y)| \leq 1.$$

Proof. When $a = \frac{2}{n}$, the Laplace transform of the $(0, a)$-generalized Fourier kernel is

$$\mathcal{L}(K^m_{\frac{2}{n}}(x, y, t)) = \Gamma(n\lambda + 1)G_1(s)G_2(s)$$

with

$$\begin{aligned} G_1(s) &= \frac{Q_{n-1}(s)}{\prod_{l=0}^{n-1}(s + inz^{1/n}\cos(\frac{q+2\pi l}{n}))}, \\ G_2(s) &= \frac{1}{(\prod_{l=0}^{n-1}(s + inz^{1/n}\cos(\frac{q+2\pi l}{n})))^\lambda}. \end{aligned}$$

Denote $g_j(t) = \mathcal{L}^{-1}(G_j), j = 1, 2$. By Lemma 6.2, we know that the inverse Laplace transform $g_2(t)$ of $G_2(s)$ is bounded by $\frac{t^{n\lambda-1}}{\Gamma(n\lambda)}$. By Theorem 6.5, we know that $g_1(t) = K^2_{\frac{2}{n}}(x, y, t)$ is bounded by 1 for any $t \in \mathbb{R}$. Using the convolution formula (6.16) again, then setting $t = 1$, we have

$$\begin{aligned} |K^m_{\frac{2}{n}}(x, y)| &= \Gamma(n\lambda + 1)\left|\int_0^1 g_1(1 - \tau)g_2(\tau)d\tau\right| \\ &\leq \Gamma(n\lambda + 1)\int_0^1 \frac{\tau^{n\lambda-1}}{\Gamma(n\lambda)}d\tau \\ &= \frac{\Gamma(n\lambda + 1)}{\Gamma(n\lambda)n\lambda} \\ &= 1. \end{aligned}$$

□

Remark 6.3. Theorem 6.9 greatly extend the applicability of the uncertainty principle and generalized translation operator in [62] and [53].

6.2.4 Integral expression of the kernel for arbitrary positive a

In Theorem 6.9, we have shown that the Fourier kernel $K^m_{\frac{2}{n}}(x, y)$ when $m \geq 2$ is the Laplace convolution of the Fourier kernel when $m = 2$ and the function $f_{n,k}(t)$ in Lemma 6.2. In this subsection we give the integral expression of the Fourier kernel of $K^m_a(x, y)$ for $m \geq 2$ and $a > 0$.

For general $a > 0$ and $m \geq 2$, the Fourier kernel in the Laplace domain can be written as

$$\begin{aligned} & \mathcal{L}(K_a^m(x,y,t)) \\ = \ & 2^{2\lambda/a}\Gamma\left(\frac{2\lambda+a}{a}\right)\frac{1}{r}\left(\frac{1}{R}\right)^{2\lambda/a}\frac{1-u_R^2}{(1-2\xi u_R+u_R^2)^{\lambda+1}} \\ = \ & 2^{2\lambda/a}\Gamma\left(\frac{2\lambda+a}{a}\right)\frac{1}{r}\left(\frac{r-s}{(\frac{2}{a}z^{a/2})^2}\right)^{2\lambda/a}\frac{1-u_R^2}{((u_R-e^{i\varrho})(u_R-e^{-i\varrho}))^{\lambda+1}}, \end{aligned}$$

where $u_R = e^{\frac{-i\pi}{a}}(\frac{2z^{a/2}}{aR})^{2/a}$, $r = \sqrt{s^2+(\frac{2}{a}z^{a/2})^2}$, $R = s+r$ and $\xi = \frac{e^{i\varrho}+e^{-i\varrho}}{2}$.

It is possible to give an integral expression of this kernel in terms of the generalized Mittag-Leffler function. We give the definition and its Laplace transform here, see also Chapter 2 in [73].

Definition 6.1. *The generalized Mittag-Leffler function is defined by*

$$E^{\delta}_{\epsilon,\gamma}(z) := \sum_{n=0}^{\infty}\frac{(\delta)_n z^n}{\Gamma(\epsilon n+\gamma)n!},$$

where $\epsilon, \gamma, \delta \in \mathbb{C}$ *with* $\operatorname{Re}\epsilon > 0$. *For* $\delta = 1$, *it reduces to the Mittag-Leffler function.*

The Laplace transform of the generalized Mittag-Leffler function is

$$\mathcal{L}(t^{\gamma-1}E^{\delta}_{\epsilon,\gamma}(bt^{\epsilon})) = \frac{1}{s^{\gamma}}\frac{1}{(1-bs^{-\epsilon})^{\delta}}$$

where $\operatorname{Re}\epsilon > 0$, $\operatorname{Re}\gamma > 0$, $\operatorname{Re}s > 0$ and $s > |b|^{1/(\operatorname{Re}\epsilon)}$, see [73].

Now, we give the integral expression of the $(0, a)$-generalized Fourier kernel as follows.

Theorem 6.10. *Let* $b_{\pm} = e^{\pm i\varrho}e^{i\pi/a}(\frac{2}{a})^{2/a}z$ *and*

$$\begin{aligned} h(t) \ = \ & z^{-2(\lambda+1)}\int_0^t \zeta^{\frac{2}{a}(\lambda+1)-1}E^{\lambda+1}_{\frac{2}{a},\frac{2}{a}(\lambda+1)}(b_+\zeta^{\frac{2}{a}}) \\ & \times(t-\zeta)^{\frac{2}{a}(\lambda+1)-1}E^{\lambda+1}_{\frac{2}{a},\frac{2}{a}(\lambda+1)}(b_-(t-\zeta)^{\frac{2}{a}})d\zeta. \end{aligned}$$

Then for $a > 0$ *and* $m \geq 2$, *the kernel of the* $(0, a)$*-generalized Fourier transform is*

$$K_a^m(x,y) \ = \ c_a^m\int_0^1\left((1+2\tau)^{-\frac{\lambda}{a}}J_{\frac{2\lambda}{a}}\left(\frac{2}{a}z^{a/2}\sqrt{1+2\tau}\right)\right.$$

$$-e^{-i\frac{2\pi}{a}}(1+2\tau)^{-\frac{\lambda+2}{a}}J_{\frac{2\lambda+4}{a}}\left(\frac{2}{a}z^{a/2}\sqrt{1+2\tau}\right)\Bigg)h(\tau)d\tau.$$

with $c_a^m = 2^{-(2\lambda+4)/a}\Gamma\left(\frac{2\lambda+a}{a}\right)e^{-i\frac{2\pi(\lambda+1)}{a}}a^{4(\lambda+1)/a}$.

Proof. Denote $\mathcal{L}(K_a^m(x,y,t)) = H_1(s)H_2(s)$ where

$$\begin{aligned} H_1(s) &= \frac{1}{(u_R - e^{i\varrho})^{\lambda+1}}\cdot\frac{1}{(u_R - e^{-i\varrho})^{\lambda+1}}, \\ H_2(s) &= 2^{2\lambda/a}\Gamma\left(\frac{2\lambda+a}{a}\right)\frac{1}{r}\left(\frac{1}{R}\right)^{2\lambda/a}(1-u_R^2). \end{aligned}$$

By direct computation, we have

$$\begin{aligned} H_1(s) &= e^{-i\frac{2\pi(\lambda+1)}{a}}\left(\left(\frac{a}{2}\right)^{2/a}z^{-1}\right)^{2(\lambda+1)} \\ &\quad \times\left[\frac{1}{(\varpi^{2/a}-b_+)^{\lambda+1}}\cdot\frac{1}{(\varpi^{2/a}-b_-)^{\lambda+1}}\right] \end{aligned}$$

with $\varpi = r - s$.

Using the generalized Mittag-Leffler function, we have

$$\mathcal{L}^{-1}\left(\frac{1}{(s^{2/a}-b)^{\lambda+1}}\right) = t^{\frac{2}{a}(\lambda+1)-1}E^{\lambda+1}_{\frac{2}{a},\frac{2}{a}(\lambda+1)}(bt^{\frac{2}{a}}). \tag{6.20}$$

Now by the inverse Laplace transform formula from [77]

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{(\sqrt{s^2+a^2}-s)^{\nu}}{\sqrt{s^2+a^2}}F(\sqrt{s^2+a^2}-s)\right) \\ &= (a^2t)^{\nu/2}\int_0^t (t+2\tau)^{-\nu/2}J_\nu(a\sqrt{t^2+2\tau t})f(\tau)d\tau \end{aligned} \tag{6.21}$$

where $\mathcal{L}(f(t)) = F(s)$, $\operatorname{Re}\nu > -1$ and $\operatorname{Re}s > |\operatorname{Im}a|$ and the Laplace convolution formula (6.16), we get the result. □

6.3 Dunkl kernel associated to the dihedral group

6.3.1 Integral expression of the kernel

The dihedral group I_k is the group of symmetries of the regular k-gon. We use complex coordinates $z_0 = x + iy$ and identify $\mathbb{R}^2$ with

$\mathbb{C}$. For a fixed k and $j \in \{0, 1, \cdots, k-1\}$, the rotations in I_k consist of $z_0 \to z_0 e^{2ij\pi/k}$ and the reflections in I_k consist of $z_0 \to \bar{z}_0 e^{2ij\pi/k}$. In particular, we have $I_1 = \mathbb{Z}_2$ and $I_2 = \mathbb{Z}_2^2$. The weight function associated with I_{2k} and $\kappa = (\alpha, \beta)$ is given by

$$\upsilon_\kappa(z_0) = \left|\frac{z_0^k - \overline{z_0}^k}{2i}\right|^{2\alpha} \left|\frac{z_0^k + \overline{z_0}^k}{2}\right|^{2\beta}.$$

The weight function $\upsilon_\kappa(z_0)$ associated with the group I_k, when k is an odd integer, is the same as the weight function $\upsilon_{(\alpha,\beta)}(z_0)$ associated with the group I_{2k} with $\beta = 0$, i.e.

$$\upsilon_\kappa(z_0) = \left|\frac{z_0^k - \overline{z_0}^k}{2i}\right|^{2\alpha}.$$

We also put $P_j(G; x, y)$ the reproducing kernel of $\mathcal{H}_j(\upsilon_\kappa)$ and by $P(G; x, y)$ the Poisson kernel, see (6.1) and (6.2). We denote by

$$d\mu_\gamma(w) = c_\gamma (1 + w)(1 - w^2)^{\gamma - 1} dw$$

with $c_\gamma = [B(\frac{1}{2}, \gamma)]^{-1}$. It was proved that finding a closed formula of the Poisson kernel which reproduces any h-harmonic in the disk reduces to the cases $k = 1$ and $k = 2$, see [38, 44].

Theorem 6.11. *[44] For each weight function $\upsilon_\kappa(z)$ associated with the group I_{2k}, the Poisson kernel is given by*

$$P(I_{2k}; z_1, z_2) = \frac{1 - |z_1|^2 |z_2|^2}{1 - |z_1^k|^2 |\overline{z_2}^k|^2} \frac{|1 - z_1^k \overline{z_2}^k|^2}{|1 - \overline{z_1} z_2|^2} P(I_2; z_1^k, z_2^k),$$

where the Poisson kernel $P(I_2; z_1, z_2)$ associated with $\upsilon_\kappa(x + iy) = |y|^{2\alpha} |x|^{2\beta}$ is given by

$$\begin{aligned} & P(I_2; z_1, z_2) \\ = & \int_{-1}^{1} \int_{-1}^{1} \frac{1 - |z_1 z_2|^2}{[1 - 2(\operatorname{Im} z_1)(\operatorname{Im} z_2) u - 2(\operatorname{Re} z_1)(\operatorname{Re} z_2) v + |z_1 z_2|^2]^{\alpha + \beta + 1}} \\ & \times d\mu_\alpha(u) d\mu_\beta(v). \end{aligned}$$

For each weight function $\upsilon_\kappa(z)$ associated with odd-k dihedral group I_k, the Poisson kernel is given by

$$P(I_k; z_1, z_2) = \frac{1 - |z_1|^2 |z_2|^2}{1 - |z_1^k|^2 |\overline{z_2}^k|^2} \frac{|1 - z_1^k \overline{z_2}^k|^2}{|1 - z_1 \overline{z_2}|^2} P(I_1; z_1^k, z_2^k)$$

where the Poisson kernel $P(I_1; z_1, z_2)$ *associated with* $\upsilon_\kappa(x+iy) = |y|^{2\alpha}$ *is given by*

$$\begin{aligned} & P(I_1; z_1, z_2) \\ = & \int_{-1}^{1} \frac{1-|z_1 z_2|^2}{(1-2(\operatorname{Im} z_1)(\operatorname{Im} z_2)u - 2(\operatorname{Re} z_1)(\operatorname{Re} z_2) + |z_1 z_2|^2)^{\alpha+1}} d\mu_\alpha(u). \end{aligned}$$

In the following, we write $z_1 = |z_1|\omega, z_2 = |z_2|\eta \in \mathbb{C}$ and $b = |z_1||z_2|$. Based on the $\mathfrak{sl}_2$ relation of $\Delta_\kappa, |x|^2$ and the Euler operator, an orthonormal basis of $L^2(\mathbb{R}^m, \upsilon_\kappa(x)dx)$ for the general Dunkl case and a series expansion of the Dunkl kernel was constructed in [4,5]. In particular, the Dunkl kernel $E_\kappa(z_1, z_2) = B_{\kappa,2}(x, y)$ associated with the dihedral group I_k has the following series expansion (see also Theorem 6.1)

$$\begin{aligned} & E_\kappa(z_1, z_2) \\ = & 2^{\langle\kappa\rangle}\Gamma(\langle\kappa\rangle+1)\sum_{j=0}^{\infty}(-i)^j b^{-\langle\kappa\rangle} J_{j+\langle\kappa\rangle}(b) P_j(I_k; \omega, \eta) \end{aligned} \quad (6.22)$$

with

$$\langle\kappa\rangle = \begin{cases} (\alpha+\beta)k/2, & \text{when } k \text{ is even;} \\ k\alpha, & \text{when } k \text{ is odd.} \end{cases}$$

We introduce an auxiliary variable t in (6.22) as follows

$$E_\kappa(z_1, z_2, t) = 2^{\langle\kappa\rangle}\Gamma(\langle\kappa\rangle+1)\sum_{j=0}^{\infty}(-i)^j b^{-\langle\kappa\rangle} J_{j+\langle\kappa\rangle}(bt) P_j(I_k; \omega, \eta).$$

Then fixing $z_1, z_2 \in \mathbb{C}$, we take the Laplace transform with respect to t. Using (6.7), $r = (s^2+b^2)^{1/2}$ and $R = s + r$, for $\operatorname{Re} s$ big enough, we have

$$\begin{aligned} \mathcal{L}(E_\kappa(z_1, z_2, t)) &= \frac{2^{\langle\kappa\rangle}\Gamma(\langle\kappa\rangle+1)}{rR^{\langle\kappa\rangle}} \sum_{j=0}^{\infty} \left(\frac{-ib}{R}\right)^j P_j(I_k; \omega, \eta) \\ &= \frac{2^{\langle\kappa\rangle}\Gamma(\langle\kappa\rangle+1)}{rR^{\langle\kappa\rangle}} P\left(I_k; \omega, \frac{-ib}{R}\eta\right) \end{aligned}$$

where $P(I_k; \omega, z_0\eta)$, $|z_0| < 1$ is the analytic continuation of the Poisson kernel $P(I_k; \omega, b\eta)$ obtained by acting with the intertwining operator V_κ on x on both sides of (6.6).

In order to get the integral expression of the Dunkl kernel, we first denote and simplify

$$
\begin{aligned}
& f_{I_{2k}}(s) \\
= \; & \frac{2^{k(\alpha+\beta)}}{rR^{k(\alpha+\beta)}} \frac{1-(\frac{-ib}{R})^2}{1-(\frac{-ib}{R})^{2k}} \frac{1-2(\frac{-ib}{R})^k \mathrm{Re}\,(\omega^k\bar{\eta}^k)+(\frac{-ib}{R})^{2k}}{1-2\mathrm{Re}\,(\omega\bar{\eta})(\frac{-ib}{R})+(\frac{-ib}{R})^2} \\
& \times \frac{1-(\frac{-ib}{R})^{2k}}{(1-2(\frac{-ib}{R})^k((\mathrm{Im}\,\omega^k)(\mathrm{Im}\,\eta^k)u+(\mathrm{Re}\,\omega^k)(\mathrm{Re}\,\eta^k)v)+(\frac{-ib}{R})^{2k})^{\alpha+\beta+1}} \\
= \; & \frac{2^{k(\alpha+\beta)}}{r} \frac{\left(R+\dfrac{b^2}{R}\right)\left(R^k-2(-ib)^k\mathrm{Re}\,(\omega^k\bar{\eta}^k)+\dfrac{(-ib)^{2k}}{R^k}\right)}{R-2(-ib)\mathrm{Re}\,(\omega\bar{\eta})+\dfrac{(-ib)^2}{R}} \\
& \times \frac{1}{\left(R^k-2(-ib)^k((\mathrm{Im}\,\omega^k)(\mathrm{Im}\,\eta^k)u+(\mathrm{Re}\,\omega^k)(\mathrm{Re}\,\eta^k)v)+\dfrac{(-ib)^{2k}}{R^k}\right)^{\alpha+\beta+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
& g_{I_k}(s) \\
= \; & \frac{2^{k\alpha}}{rR^{k\alpha}} \frac{1-(\frac{-ib}{R})^2}{1-(\frac{-ib}{R})^{2k}} \frac{1-2(\frac{-ib}{R})^k \mathrm{Re}\,(\omega^k\bar{\eta}^k)+(\frac{-ib}{R})^{2k}}{1-2\mathrm{Re}\,(\omega\bar{\eta})(\frac{-ib}{R})+(\frac{-ib}{R})^2} \\
& \times \frac{1-(\frac{-ib}{R})^{2k}}{(1-2(\frac{-ib}{R})^k((\mathrm{Im}\,\omega^k)(\mathrm{Im}\,\eta^k)u+(\mathrm{Re}\,\omega^k)(\mathrm{Re}\,\eta^k))+(\frac{-ib}{R})^{2k})^{\alpha+1}} \\
= \; & \frac{2^{k\alpha}}{r} \frac{\left(R+\dfrac{b^2}{R}\right)\left(R^k-2(-ib)^k\mathrm{Re}\,(\omega^k\bar{\eta}^k)+\dfrac{(-ib)^{2k}}{R^k}\right)}{R-2(-ib)\mathrm{Re}\,(\omega\bar{\eta})+\dfrac{(-ib)^2}{R}} \\
& \times \frac{1}{\left(R^k-2(-ib)^k((\mathrm{Im}\,\omega^k)(\mathrm{Im}\,\eta^k)u+(\mathrm{Re}\,\omega^k)(\mathrm{Re}\,\eta^k))+\dfrac{(-ib)^{2k}}{R^k}\right)^{\alpha+1}}.
\end{aligned}
$$

By $R=s+r=s+\sqrt{s^2+b^2}$ and $\dfrac{1}{R}=\dfrac{1}{s+\sqrt{s^2+b^2}}=\dfrac{\sqrt{s^2+b^2}-s}{b^2}$, we get

$$
\begin{aligned}
R+\frac{b^2}{R} &= s+r+b^2\frac{r-s}{b^2}=2r \\
R+\frac{(-ib)^2}{R} &= s+r-b^2\frac{r-s}{b^2}=2s
\end{aligned}
$$

and

$$R^k + \frac{(-ib)^{2k}}{R^k} = (s+r)^k + (-1)^k(r-s)^k = \sum_{j=0}^{k} \binom{k}{j}(1+(-1)^{k+j})s^j r^{k-j}$$

which means that $R^k + \frac{(-ib)^{2k}}{R^k}$ is always a polynomial in s as k is a positive integer. We can apply Lemma 6.1 because $|(\operatorname{Im}\omega^k)(\operatorname{Im}\eta^k)u + (\operatorname{Re}\omega^k)(\operatorname{Re}\eta^k)v| \leq 1$, for $u, v \in [-1, 1]$. Hence $f_{I_{2k}}(s)$ and $g_{I_k}(s)$ have the following factorization,

Lemma 6.3. *Let*

$$\begin{aligned} A(s,q) &= \prod_{l=0}^{k-1}\left(s + ib\cos\left(\frac{q+2\pi l}{k}\right)\right), \\ B(s) &= (s + ib\mathrm{Re}\,(\omega\bar{\eta})). \end{aligned}$$

Then $f_{I_{2k}}(s)$ has the following factorization

$$\begin{aligned} & f_{I_{2k}}(s) \\ = & \frac{A(s,q(1,1))}{B(s)[A(s,q(u,v))]^{\alpha+\beta+1}} \\ = & \frac{1}{B(s)[A(s,q(u,v))]^{\alpha+\beta}} + \frac{(-ib)^k\cos(q(u-1,v-1))}{2^{k-1}B(s)[A(s,q(u,v))]^{\alpha+\beta+1}}, \end{aligned}$$

and $g_{I_k}(s)$ has the following factorization

$$\begin{aligned} & g_{I_k}(s) \\ = & \frac{A(s,q(1,1))}{B(s)[A(s,q(u,1))]^{\alpha+1}} \\ = & \frac{1}{B(s)[A(s,q(u,1))]^{\alpha}} + \frac{(-ib)^k\cos(q(u-1,0))}{2^{k-1}B(s)[A(s,q(u,1))]^{\alpha+1}}. \end{aligned}$$

where $q(u,v) = \arccos((\operatorname{Im}\omega^k)(\operatorname{Im}\eta^k)u + (\operatorname{Re}\omega^k)(\operatorname{Re}\eta^k)v)$.

Proof. For the first equality, we only need to show that $q(1,1) = \arccos(\mathrm{Re}\,(\omega^k\bar{\eta}^k))$, i.e.

$$\mathrm{Re}\,(\omega^k\bar{\eta}^k) = (\operatorname{Im}\omega^k)(\operatorname{Im}\eta^k) + (\operatorname{Re}\omega^k)(\operatorname{Re}\eta^k)$$

which follows by expanding the left-hand side. For the second equality, we have used

$$2^k A(s,q(u,v))$$

$$= \quad R^k - 2(-ib)^k((\operatorname{Im}\omega^k)(\operatorname{Im}\eta^k)u + (\operatorname{Re}\omega^k)(\operatorname{Re}\eta^k)) + \frac{(-ib)^{2k}}{R^k}.$$

□

Now, we have our first main result in this section

Theorem 6.12. *For the even dihedral group I_{2k}, the radial Laplace transform of the Dunkl kernel is*

$$\mathcal{L}(E_\kappa(z_1,z_2,t)) \quad = \quad \Gamma(k(\alpha+\beta)+1)\int_{-1}^{1}\int_{-1}^{1} f_{I_{2k}}(s)d\mu_\alpha(u)d\mu_\beta(v).$$

For odd-k dihedral group I_k, the radial Laplace transform of the Dunkl kernel $E_\kappa(z_1,z_2,t)$ is

$$\mathcal{L}(E_\kappa(z_1,z_2,t)) \quad = \quad \Gamma(k\alpha+1)\int_{-1}^{1} g_{I_k}(s)d\mu_\alpha(u).$$

For any dihedral group, when the multiplicity function κ takes integer values, we know from Lemma 6.3 that $f_{I_{2k}}(s)$ and $g_{I_k}(s)$ are rational functions. So then the Dunkl kernel can be obtained by the inverse Laplace transform through partial fraction decomposition using Theorem 6.12 and 6.7.

Remark 6.4. It is known that the Dunkl kernel for positive integer κ can in principle be expressed as elementary functions, see [75] and [31]. However, this is not made concrete there. In [33], the authors use the shift principle of [75] and act with multiple combinations of the Dunkl operators on the Dunkl Bessel function to derive the Dunkl kernel in the dihedral setting. However, there the Dunkl Bessel function was only known in a few cases. In subsection 4.2, we will give the integral expression of the generalized Bessel function using the Laplace transform. Also, acting multiple combinations of the Dunkl operators turns out not to be feasible in practice.

When the multiplicity function κ is not integer valued, we can still derive integral formulas for the kernel using Theorem 6.12. First denote

$$\begin{aligned} & g_\alpha(t,q(u,v)) \\ = \quad & \mathcal{L}^{-1}\left(\frac{1}{A(s,q(u,v))^\alpha}\right) = \mathcal{L}^{-1}\left(\frac{1}{\prod_{l=0}^{k-1}\left(s+ib\cos\left(\frac{q(u,v)+2\pi l}{k}\right)\right)^\alpha}\right) \end{aligned}$$

$$= \mathcal{L}^{-1}\left(\frac{2^{k\alpha}}{\left(R^k - 2(-ib)^k((\operatorname{Im}\omega^k)(\operatorname{Im}\eta^k)u + (\operatorname{Re}\omega^k)(\operatorname{Re}\eta^k)v) + \frac{(-ib)^{2k}}{R^k}\right)^\alpha}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{2^{k\alpha-1}e^{ik\alpha\pi}\varpi_0^{k\alpha}}{r}\left(\frac{b^2}{\varpi_0}+\varpi_0\right)\right.$$
$$\left.\times\frac{1}{(\varpi_0^k - e^{iq(u,v)}(e^{i\frac{\pi}{2}}b)^k)^\alpha(\varpi_0^k - e^{-iq(u,v)}(e^{i\frac{\pi}{2}}b)^k)^\alpha}\right)$$

where $\varpi_0 = r - s$. Using the same method as in Theorem 6.10, by formula (6.20) and (6.21), we have

$$\begin{aligned} & g_\alpha(t, q(u,v)) \\ = \ & 2^{k\alpha-1}e^{ik\alpha\pi}b^{k\alpha+1}\int_0^t \Big[J_{k\alpha-1}(b\sqrt{t^2-2\tau t}) \\ & +t(t+2\tau)^{-1}J_{k\alpha+1}(b\sqrt{t^2+2\tau t})\Big]t^{\frac{k\alpha-1}{2}}(t+2\tau)^{-\frac{k\alpha-1}{2}}\tilde{h}_\alpha(\tau)d\tau, \end{aligned}$$

where $\tilde{h}_\alpha(t)$ is the convolution of two generalized Mittag-Leffler functions,

$$\begin{aligned} \tilde{h}_\alpha(t) \ = \ & \int_0^t \zeta^{k\alpha-1}E^\alpha_{k,k\alpha}(e^{iq(u,v)}(e^{i\frac{\pi}{2}}b)^k\zeta^k)(t-\zeta)^{k\alpha-1} \\ & \times E^\alpha_{k,k\alpha}(e^{-iq(u,v)}(e^{i\frac{\pi}{2}}b)^k(t-\zeta)^k)d\zeta. \end{aligned}$$

Now, by the convolution formula (6.16), we have

Theorem 6.13. *Let $a^l_{u,v}$ be the $k+1$ roots of $B(s)A(s,q(u,v))$, i.e. $a^l_{u,v} = -ib\cos\left(\frac{q+2\pi l}{k}\right)$, $l = 0,\cdots,k-1$ and $a^k_{u,v} = -ib\mathrm{Re}\,(\omega\bar{\eta})$. Then for each dihedral group I_{2k} and positive multiplicity function κ, the Dunkl kernel is given by*

$$\begin{aligned} & E_\kappa(z_1,z_2) \\ = \ & \int_{-1}^1\int_{-1}^1\int_0^1\left[\sum_{l=0}^k \frac{A(s,q(1,1))(s-a^l_{u,v})}{B(s)A(s,q(u,v))}\Big|_{s=a^l_{u,v}} e^{a^l_{u,v}(1-\tau)}\right] \\ & \times\Gamma(k(\alpha+\beta)+1)g_{\alpha+\beta}(\tau,q(u,v))d\tau d\mu_\alpha(u)d\mu_\beta(v). \end{aligned}$$

For each odd-k dihedral group I_k and positive multiplicity function κ, the Dunkl kernel is

$$E_\kappa(z_1,z_2)$$

$$
\begin{aligned}
&= \Gamma(k\alpha+1)\int_{-1}^{1}\int_{0}^{1}\left[\sum_{l=0}^{k}\frac{A(s,q(1,1)(s-a_{u,1}^{l}))}{B(s)A(s,q(u,1))}\bigg|_{s=a_{u,1}^{l}} e^{a_{u,1}^{l}(1-\tau)}\right] \\
&\quad \times g_\alpha(\tau,q(u,1))d\tau d\mu_\alpha(u),
\end{aligned}
$$

where $q(u,v)=\arccos((\operatorname{Im}\omega^k)(\operatorname{Im}\eta^k)u+(\operatorname{Re}\omega^k)(\operatorname{Re}\eta^k)v)$.

Proof. We only prove the odd dihedral group I_k cases. We write g_{I_k} as

$$
g_{I_k}(s)=\frac{A(s,q(1,1))}{B(s)[A(s,q(u,1))]}\cdot\frac{1}{[A(s,q(u,1))]^\alpha}. \tag{6.23}
$$

The inverse Laplace transform of the second factor on the right-hand side of (6.23) is $g_\alpha(t,q(u,1))$. The first factor on the right-hand side of (6.23) is inversed by partial fraction decomposition. Then by the Laplace convolution formula (6.16), we get the result.

□

Using the second equality in Lemma 6.3, the integral expression of the Dunkl kernel also reduces to the integral expression of $f_{n,k}(t)$ in Lemma 6.2. Indeed, put

$$
\begin{aligned}
& h_\alpha(t,q(u,v)) \\
&= \mathcal{L}^{-1}\left(\frac{1}{B(s)A(s,q(u,v))^\alpha}\right) \\
&= \mathcal{L}^{-1}\left(\frac{1}{(s+ib\mathrm{Re}\,(\omega\bar\eta))\prod_{l=0}^{k-1}\left(s+ib\cos\left(\frac{q(u,v)+2\pi l}{k}\right)\right)^\alpha}\right),
\end{aligned}
$$

which is the convolution of $g_\alpha(t,q(u,v))$ and $e^{-ib\mathrm{Re}\,(\omega\bar\eta)}$. Then we have

Theorem 6.14. *For each dihedral group I_{2k} and positive multiplicity function κ, the Dunkl kernel is given by*

$$
\begin{aligned}
& E_\kappa(z_1,z_2) \\
&= \Gamma(k(\alpha+\beta)+1)\int_{-1}^{1}\int_{-1}^{1}[h_{\alpha+\beta}(1,q(u,v))+2^{1-k}(-ib)^k \\
&\quad \times\cos(q(u-1,v-1))h_{\alpha+\beta+1}(1,q(u,v))]d\mu_\alpha(u)d\mu_\beta(v).
\end{aligned}
$$

For each odd-k dihedral group I_k and positive multiplicity function κ, the Dunkl kernel is

$$\begin{aligned} & E_\kappa(z_1, z_2) \\ = \; & \Gamma(k\alpha+1)\int_{-1}^{1}[h_\alpha(1, q(u,1)) + 2^{1-k}(-ib)^k \\ & \times \cos(q(u-1,0))h_{\alpha+1}(1, q(u,1))]d\mu_\alpha(u), \end{aligned}$$

where $q(u,v) = \arccos((\operatorname{Im}\omega^k)(\operatorname{Im}\eta^k)u + (\operatorname{Re}\omega^k)(\operatorname{Re}\eta^k)v)$.

Let us now look at a few special cases. When $k = 1$ and any positive α, $g_{I_1}(s)$ becomes

$$g_{I_1}(s) \;=\; \frac{1}{(s + ib((\operatorname{Im}\omega)(\operatorname{Im}\eta)u + (\operatorname{Re}\omega)(\operatorname{Re}\eta)))^{\alpha+1}}. \tag{6.24}$$

We take the inverse Laplace transform of (6.24) and set $t = 1$, then we reobtain the Dunkl kernel for I_1, which is

$$E_\kappa(z_1, z_2) = \int_{-1}^{1} e^{-i(u\operatorname{Im} z_1 \operatorname{Im} z_2 + \operatorname{Re} z_1 \operatorname{Re} z_2)} d\mu_\alpha(u).$$

It coincides with the known result of the integral representation of the intertwining operator of the rank 1 case, for $\operatorname{Re}\alpha > 0$,

$$V_\alpha p(x) = \int_{-1}^{1} p(xu)d\mu_\alpha(u),$$

which can be found in [44]. Similarly, we reobtain the Dunkl kernel for I_2, which is

$$E_\kappa(z_1, z_2) = \int_{-1}^{1}\int_{-1}^{1} e^{-i(u\operatorname{Im} z_1 \operatorname{Im} z_2 + v\operatorname{Re} z_1 \operatorname{Re} z_2)} d\mu_\alpha(u)d\mu_\beta(v)$$

which coincides with the result obtained using the intertwining operator for $\mathbb{Z}_2^2$.

For the dihedral group I_3 and I_6, we can get the integral expression of the Dunkl kernels by (6.19) as both of them are related to the function $f_{3,k}(t)$.

For the dihedral group I_4, we have

$$f_{I_4}(s)$$

$$= \frac{s^2 + b^2\left(\frac{1+\mathrm{Re}\,\omega^2\bar{\eta}^2}{2}\right)}{(s+ib\mathrm{Re}\,\omega\bar{\eta})\left(s^2+b^2\left(\frac{1+(\mathrm{Im}\,\omega^2)(\mathrm{Im}\,\eta^2)u+(\mathrm{Re}\,\omega^2)(\mathrm{Re}\,\eta^2)v}{2}\right)\right)^{\alpha+\beta+1}}.$$

We take the inverse Laplace transform and set $t = 1$. We get the Dunkl kernel for I_4, using Theorem 6.13,

$$\begin{aligned} &E_\kappa(z_1, z_2) \\ = \; &\frac{\sqrt{\pi}\Gamma(2(\alpha+\beta)+1)}{2^{\alpha+\beta-1/2}\Gamma(\alpha+\beta)} \int_{-1}^1\int_{-1}^1\int_0^1 \frac{1}{\theta_2^2-\theta_3^2}\Bigg(e^{-ib\theta_3\tau}(\theta_1^2-\theta_3^2) \\ &+(\theta_1^2-\theta_2^2)\left(\frac{i\theta_3}{\theta_2}\sin(b\theta_2\tau)-\cos(b\theta_2\tau)\right)\Bigg) \\ &(1-\tau)^{\alpha+\beta-1/2}\frac{J_{\alpha+\beta-1/2}(b\theta_2(1-\tau))}{(b\theta_2)^{\alpha+\beta-1/2}}d\tau d\mu_\alpha(u)d\mu_\beta(v), \end{aligned}$$

or using Theorem 6.14,

$$\begin{aligned} &E_\kappa(z_1, z_2) \\ = \; &\frac{\sqrt{\pi}\Gamma(2(\alpha+\beta)+1)}{2^{\alpha+\beta-1/2}\Gamma(\alpha+\beta)}c_\alpha c_\beta \int_{-1}^1\int_{-1}^1\int_0^1 e^{-ib\theta_3\tau} \\ &\times\Bigg((1-\tau)^{\alpha+\beta-1/2}\frac{J_{\alpha+\beta-1/2}(b\theta_2(1-\tau))}{(b\theta_2)^{\alpha+\beta-1/2}} + \frac{b^2(\theta_1^2-\theta_2^2)}{2(\alpha+\beta)} \\ &\times(1-\tau)^{\alpha+\beta+1/2}\frac{J_{\alpha+\beta+1/2}(b\theta_2(1-\tau))}{(b\theta_2)^{\alpha+\beta+1/2}}\Bigg)d\tau d\mu_\alpha(u)d\mu_\beta(v), \end{aligned}$$

where $\theta_1 = \sqrt{\frac{1+(\mathrm{Re}\,\omega^2\bar{\eta}^2)}{2}}$, $\theta_2 = \sqrt{\frac{1+(\mathrm{Im}\,\omega^2)(\mathrm{Im}\,\eta^2)u+(\mathrm{Re}\,\omega^2)(\mathrm{Re}\,\eta^2)v}{2}}$ and $\theta_3 = \mathrm{Re}\,\omega\bar{\eta}$.

Remark 6.5. The kernel of the (κ, a)-generalized Fourier transform with dihedral symmetry can be obtained similarly.

6.3.2 Dunkl Bessel function

Following [44], we define the Dunkl Bessel function by

$$D_\kappa(z_1, z_2) = \frac{1}{|I_k|}\sum_{g\in I_k} E_\kappa(z_1, g\cdot z_2).$$

Let $z_1 = |z_1|e^{i\phi_1}$, $z_2 = |z_2|e^{i\phi_2}$, $\phi_1, \phi_2 \in [1, \pi/2k]$ and $b = |z_1||z_2|$. Then the Dunkl Bessel function associated to $I_{2k}, k \geq 2$ is given by (see [35])

$$\begin{aligned} & D_\kappa(|z_1|, \phi_1, |z_2|, \phi_2) \\ = \quad & c_{k,\kappa} \left(\frac{2}{b}\right)^{\langle\kappa\rangle} \sum_{j=0}^{\infty} i^{2kj+\langle\kappa\rangle} J_{2kj+\langle\kappa\rangle}(b) \\ & \times p_j^{\alpha-1/2,\beta-1/2}(\cos(2k\phi_1)) p_j^{\alpha-1/2,\beta-1/2}(\cos(2k\phi_2)) \end{aligned}$$

where $p_j^{\alpha-1/2,\beta-1/2}$ is the j-th orthonormal Jacobi polynomial of parameters $(\alpha - 1/2, \beta - 1/2)$ and

$$c_{k,\kappa} = 2^{\alpha+\beta} \frac{\Gamma(\langle\kappa\rangle + 1)\Gamma(\alpha + 1/2)\Gamma(\beta + 1/2)}{\Gamma(\alpha + \beta + 1)}.$$

With the Dijksma-Koornwinder product formula for the Jacobi polynomial [35], the Dunkl Bessel function becomes

$$\begin{aligned} & D_\kappa(|z_1|, \phi_1, |z_2|, \phi_2) \\ = \quad & \Gamma(\langle\kappa\rangle + 1) \int_{-1}^{1} \int_{-1}^{1} \left(\frac{2}{b}\right)^{\langle\kappa\rangle} \sum_{j=0}^{\infty} \frac{(2j + \alpha + \beta)}{\alpha + \beta} i^{2kj+\langle\kappa\rangle} J_{2kj+\langle\kappa\rangle}(b) \\ & \times C_{2j}^{\alpha+\beta}(z_{k\phi_1,k\phi_2}(u, v)) \mu^\alpha(du) \mu^\beta(dv) \end{aligned} \tag{6.25}$$

where μ^α is the symmetric beta probability measure

$$\mu^\alpha(du) = \frac{\Gamma(\alpha + 1/2)}{\sqrt{\pi}\Gamma(\alpha)} (1 - u^2)^{\alpha-1} du, \quad \alpha > -1,$$

and

$$z_{\phi_1,\phi_2}(u, v) = u \cos\phi_1 \cos\phi_2 + v \sin\phi_1 \sin\phi_2,$$

and $C_{2j}^\alpha(x)$ the Gegenbauer polynomial. Now the integrand of (6.25) equals $\dfrac{f_{2k}^+ + f_{2k}^-}{2}$ with

$$\begin{aligned} f_{2k}^\pm(b, \xi) \quad = \quad & \Gamma(k(\alpha + \beta) + 1) \left(\frac{2}{b}\right)^{k(\alpha+\beta)} \sum_{j=0}^{\infty} \frac{(j + \alpha + \beta)}{\alpha + \beta} (\pm 1)^j \\ & \times e^{i\frac{\pi}{2}k(j+\alpha+\beta)} J_{k(j+\alpha+\beta)}(b) C_j^{\alpha+\beta}(z_{k\phi_1,k\phi_2}). \end{aligned}$$

As before, we introduce an auxiliary variable t in the series

$$\begin{aligned} f_{2k}^{\pm}(b,\xi,t) &= \Gamma(k(\alpha+\beta)+1)\left(\frac{2}{b}\right)^{k(\alpha+\beta)} \sum_{j=0}^{\infty} \frac{(j+\alpha+\beta)}{\alpha+\beta}(\pm 1)^j \\ &\quad \times e^{i\frac{\pi}{2}k(j+\alpha+\beta)} J_{k(j+\alpha+\beta)}(bt) C_j^{\alpha+\beta}(z_{k\phi_1,k\phi_2}). \end{aligned}$$

and take the Laplace transform term by term. This yields

$$\begin{aligned} \mathcal{L}(f_{2k}^{\pm}) &= \Gamma(k(\alpha+\beta)+1)\frac{(2e^{i\frac{\pi}{2}})^{k(\alpha+\beta)}}{r} \\ &\quad \times \frac{R^k - \frac{(-1)^k b^{2k}}{R^k}}{(R^k - 2(\pm(ib)^k) z_{k\phi_1,k\phi_2} + \frac{(-1)^k b^{2k}}{R^k})^{\alpha+\beta+1}} \\ &= \Gamma(k(\alpha+\beta)+1)\frac{(2e^{i\frac{\pi}{2}})^{k(\alpha+\beta)}}{r} \\ &\quad \times \frac{(r+s)^k - (-1)^k (r-s)^k}{((r+s)^k - 2(\pm(ib)^k) z_{k\phi_1,k\phi_2} + (-1)^k (r-s)^k)^{\alpha+\beta+1}} \end{aligned} \tag{6.26}$$

where $r = \sqrt{r^2+b^2}$, $R = s + r$. Comparing (6.26) with (6.11), and using the same method as in Theorem 6.9, we get $|f_{2k}^{\pm}| \leq 1$. Then we have

$$|D_\kappa(z_1,z_2)| = \left| \int_{-1}^{1}\int_{-1}^{1} \frac{f_{2k}^{+} + f_{2k}^{-}}{2} \mu^\alpha(du)\mu^\beta(dv) \right| \leq 1$$

because $\int_{-1}^{1}\int_{-1}^{1} \mu^\alpha(du)\mu^\beta(dv) = 1$, giving an alternative and direct proof of the boundedness of the Dunkl Bessel function. Also, using (6.26) and (6.11), it is now in principle possible to find an integral expression for the Dunkl Bessel function. We illustrate this for the dihedral group I_4. In that case, we have

$$\begin{aligned} \mathcal{L}(f_4^{\pm}) &= \Gamma(2(\alpha+\beta)+1)\frac{(2e^{i\frac{\pi}{2}})^{2(\alpha+\beta)}}{r} \\ &\quad \times \frac{(r+s)^2 - (-1)^2 (r-s)^2}{((r+s)^2 - 2(\pm(ib)^2) z_{2\phi_1,2\phi_2} + (-1)^2 (r-s)^2)^{\alpha+\beta+1}} \\ &= \Gamma(2(\alpha+\beta)+1) e^{i\pi(\alpha+\beta)} \frac{s}{\left(s^2 + b^2\left(\frac{1-(\pm z_{2\phi_1,2\phi_2})}{2}\right)\right)^{\alpha+\beta+1}}. \end{aligned}$$

Using the inverse Laplace transform formula (6.10), we have, after evaluating at $t = 1$,

$$\begin{aligned}
& f_4^+ + f_4^- \\
= \; & e^{i\pi(\alpha+\beta)} \frac{\sqrt{\pi}\Gamma(2(\alpha+\beta)+1)}{\Gamma(\alpha+\beta+1)2^{\alpha+\beta+1/2}} \left(\frac{J_{\alpha+\beta-1/2}(b_1)}{b_1^{\alpha+\beta-1/2}} + \frac{J_{\alpha+\beta-1/2}(b_2)}{b_2^{\alpha+\beta-1/2}} \right) \\
= \; & e^{i\pi(\alpha+\beta)} 2^{\alpha+\beta-1/2} \Gamma(\alpha+\beta+\frac{1}{2}) \left(\frac{J_{\alpha+\beta-1/2}(b_1)}{b_1^{\alpha+\beta-1/2}} + \frac{J_{\alpha+\beta-1/2}(b_2)}{b_2^{\alpha+\beta-1/2}} \right)
\end{aligned}$$

where $b_1 = \left(b\sqrt{\frac{1-z_{2\phi_1,2\phi_2}}{2}} \right)$, $b_2 = \left(b\sqrt{\frac{1+z_{2\phi_1,2\phi_2}}{2}} \right)$. In the second equality, we have used the Gauss duplication formula

$$\sqrt{\pi}\Gamma(2v) = 2^{2v-1}\Gamma(v)\Gamma(v+1/2).$$

Hence for I_4, the Dunkl Bessel function is given by

$$\begin{aligned}
& D_\kappa(z_1, z_2) \\
= \; & e^{i\pi(\alpha+\beta)} 2^{\alpha+\beta-3/2} \Gamma(\alpha+\beta+1/2) \\
& \times \int_{-1}^{1}\int_{-1}^{1} \left(\frac{J_{\alpha+\beta-1/2}(b_1)}{b_1^{\alpha+\beta-1/2}} + \frac{J_{\alpha+\beta-1/2}(b_2)}{b_2^{\alpha+\beta-1/2}} \right) \mu^\alpha(du)\mu^\beta(dv).
\end{aligned}$$

Remark 6.6. When $\alpha + \beta$ is integer, the integral expression of the Dunkl Bessel function associated to I_4 was obtained in [35]. Our result hence extends this result to arbitrary $\alpha, \beta > 0$.

Remark 6.7. For odd dihedral groups, the integral expression of the Dunkl Bessel function is computed in a similar way.

It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power.

Paul Adrien Maurice Dirac

7 Conclusions and open problems

This dissertation is mainly devoted to develop the Laplace domain technique to obtain the closed expression of the hyper-complex Fourier kernel. The reason why the Laplace method works is that the radial Laplace transform builds a bridge from the Fourier kernel to the Poisson kernel. In the hyper-complex setting, the Poisson kernel further breaks into the Cauchy kernel and the Szegö kernel. With this relation, we get the closed expression of the Fourier kernel in the Laplace domain using the monogenic expansion. The closed expression of even dimensional kernels and the integral expression for all dimension are obtained by the Laplace inversion. For the Clifford-Fourier transform, this was achieved in Chapter 3 and the generalized Clifford-Fourier kernel in Chapter 4. The Clifford-Fourier transform on hyperbolic space was developed in Chapter 5 using the results in the Laplace domain. This method finally leads also to the integral expression of the (κ, a)-generalized Fourier kernel and the Dunkl dihedral kernel in Chapter 6.

We have recovered the known results of the even dimensional Clifford-Fourier kernel and given a new integral expression for all dimensions in Chapter 3. However, the bound of the odd dimensional kernel is still open. The bound is of great importance to further develop the analytic properties. The same problem exists in the generalized Clifford-Fourier transform setting. Some estimate of the kernel from the integral expression or other new methods would yield a new breakthrough. Alternatively, as the bound of the even dimensional kernel is known, one possible future research problem is on the analytic properties of this kind of Fourier transform. Also, it will be interesting to give an application of this transform in image processing.

In Chapter 5, we introduced the integral transform on the hyperbolic space. The hyperbolic space has very rich structure. However the study of the hyper-complex integral transform on the hyperbolic space is only at its beginning. It will be interesting to develop a hyper-complex wavelets on the hyperbolic space.

In Chapter 6, we have given the integral expression of the Dunkl dihedral kernel using the explicit expression of the Poisson kernel. The expressions of other Dunkl kernels are still open. Another very interesting and important problem is how to construct the intertwining operator explicitly. One way to solve this problem is to establish the connection of the Dunkl kernel and the Rösler measure [81]. At present, a more solvable problem is to obtain the Laplace type expression of the Dunkl kernel from our Laplace domain result. For the radially deformed Fourier transform, we computed the bounds successfully for $a = \frac{2}{n}$. The bounds for other cases is still open and has been the obstacle to develop further analysis. Note that the hypercomplex Fourier transform in the Dunkl setting has been introduced in [27]. Besides the closed expression and the bounds, the generalized translation and convolution in these cases also deserves closely study. The operator of the Clifford Hermite semigroup also deserves deep study.

English Summary

Since its introduction by Fourier in the early 1800s, the Fourier transform has found innumerable applications in sciences and engineering. In recent years, many efforts have been devoted to study and experiment with hyper-complex Fourier transforms for use in image processing. The main feature of these Fourier transforms is the representation of a multichannel signal as a pure quaternion or as an element of a suitable Clifford algebra. At present, in the literature, there are mainly 3 approaches to hyper-complex Fourier transforms considered, we refer to the review papers [16] and [23].

In this thesis, we consider the hyper-complex Fourier transform defined by prescribing eigenvalues to a suitable basis of a Clifford-algebra valued L^2 space which is called the eigenfunction approach in the literature. This kind of integral transforms is mainly studied in [13–15] [29] [73] and [28]. These transforms have a deep connection with quantum mechanics and exhibit a very particular underlying algebraic structure, namely that of the Lie superalgebra $\mathfrak{osp}(1|2)$. With this algebraic structure, one can see that these transforms satisfy many important properties of the classical Fourier transform, such as the Helmholtz relations, inversion, Plancherel theorem and uncertainty principle, etc. One of the main problems in this approach is to determine the integral kernel explicitly. At present, there are only few methods to compute it. The main goal of this thesis is to develop a new method to compute the kernels of these Fourier transforms and their generalization on the hyperbolic space.

Other hyper-complex Fourier theory related with the eigenfunction approach is based on the Dunkl operators, we refer to [27] for these generalizations. The Dunkl operator was introduced by C.F. Dunkl in the late 80ies. Now it has become the key tool in the study of special functions with reflection symmetries, see [44]. The explicit computation of the joint eigenfunction of the Dunkl operators is a difficult problem, even more so for its generalization in the radially

deformed case or the Clifford deformed case. In this thesis, we develop the Laplace method further to give the explicit expression of the Dunkl kernel in the dihedral setting. The computation of the generalized kernel in the Clifford case is similar.

In the following, we give an overview of the contents of this thesis.

In Chapter 2, we give the preliminaries for this thesis. It consists of 4 subsections, namely the Clifford algebra, the fundamental facts of Euclidean Clifford analysis, the Laplace transform which is the main tool, and the operator exponential in the classical Fourier analysis.

In Chapter 3, we introduce the Laplace transform method to compute the Clifford-Fourier kernel. This is done by introducing an auxiliary variable t and subsequently expressing the classical Fourier kernel by the Cauchy kernel and the Szegö kernel in the Laplace domain. The action of the exponential operator is understood using the monogenic expansion of the two kernels. We get the closed expression of the even dimensional kernels by Laplace inversions. Moreover, we are able to compute the generating function of the even dimensional kernels. The new method also recovers the plane wave expansion of the kernels given in the previous method. For the odd dimension, now an integral formula is obtained.

In Chapter 4, we further develop the Laplace transform method to compute the generalized Fourier kernel obtained in [28] using the representation theory of $\mathfrak{sl}_2$. We first establish the connection between the kernel of the Clifford-Fourier transform and the generalized Clifford-Fourier transform. From that relationship, we give the explicit expression of the kernel and the generating function of the even dimensional kernels. Furthermore, we determine polynomials G which give rise to polynomially bounded kernels. This is interesting because it offers new perspectives to define odd dimensional hypercomplex Fourier transform with proper bounded kernel functions.

In Chapter 5, we concentrate on developing the Clifford-Fourier transform on hyperbolic space. The representation and analysis of signals in non-Euclidean geometry is now a recurrent problem in many scientific domains. A lot of efforts have been devoted to this problem, see $[1, 2, 10]$. Because the geodesic sphere in H^m is a Euclidean sphere, the spherical Dirac operator will play the same role in defining the generalized Fourier transform both on the Euclidean space and the hyperbolic space. The new transform is defined by acting with the exponential of the spherical Dirac operator on the Helgason Fourier kernel. Based on the connection between the Laplace domain results of the Euclidean case and the hyperbolic case, we get the explicit expression of all even dimensional kernels.

In Chapter 6, we develop the Laplace method to obtain explicit and integral expressions for the kernel of the (κ, a)-generalized Fourier transform for $\kappa = 0$. In the case of dihedral groups, this method is also applied to the Dunkl kernel as well as the Dunkl Bessel function. By making use of the Poisson kernel, the kernel in the Laplace domain takes on a much simpler form. The inverse Laplace transform is then computed using the generalized Mittag-Leffler function to obtain integral expressions. In case the parameters involved are integers, explicit formulas will be obtained using partial fraction decomposition. New bounds for the kernel of the (κ, a)- generalized Fourier transform are obtained as well.

Dutch Summary

Sinds zij door Fourier in de vroege jaren 1800 werd ingevoerd, heeft de Fouriertransformatie talloze toepassingen in de wetenschappen en techniek gevonden. Tijdens de laatste jaren zijn veel pogingen ondernomen om hypercomplexe Fouriertransformaties in te voeren en te bestuderen voor beeldverwerking. Het belangrijkste kenmerk van deze Fouriertransformaties is de voorstelling van een multikanaalsignaal als een zuiver quaternion of als een element van een geschikte Cliffordalgebra. Momenteel zijn er in de literatuur hiervoor drie hoofdbenaderingen van hypercomplexe Fouriertransformaties onderzocht, we verwijzen hiervoor naar de overzichtsartikelen [16] en [23].

In dit proefschrift beschouwen we de hypercomplexe Fouriertransformatie gedefinieerd door het voorschrijven van eigenwaarden voor een geschikte basis van een Clifford-algebrawaardige ruimte, hetgeen in de literatuur de eigenfunctieaanpak wordt genoemd. Dit soort integraaltransformaties werden voornamelijk bestudeerd in [13–15] [29] [73] en [28]. Deze transformaties houden diep verband met de kwantummechanica en vertonen een zeer bijzondere onderliggende algebraïsche structuur, namelijk die van de Lie superalgebra $\mathfrak{osp}(1|2)$. Dankzij deze algebraïsche structuur kan men inzien waarom deze transformaties voldoen aan een aantal belangrijke eigenschappen van de klassieke Fouriertransformatie, zoals de Helmholtzrelaties, de inversie, de stelling van Plancherel en het onzekerheidsprincipe, etc.

Eén van de belangrijkste problemen bij deze benadering is het expliciet bepalen van de integraalkern; er zijn tot op heden maar weinig manieren om deze nu te berekenen. Het belangrijkste doel van dit proefschrift is het ontwikkelen van een nieuwe methode om de kernen van deze Fouriertransformaties en hun veralgemening tot de hyperbolische ruimte te berekenen.

Een andere hypercomplexe Fouriertheorie in verband met de eigenfunctiebenadering is gebaseerd op de Dunkloperatoren, zie [27] voor deze veralgemeningen. De Dunkloperator werd geïntroduceerd door

C.F. Dunkl in de late jaren 1980. Nu is deze uitgegroeid tot het belangrijkste instrument in de studie van speciale functies met spiegelsymmetrieën, zie [44]. De expliciete berekening van de gemeenschappelijke eigenfuncties van de Dunkloperatoren is evenwel een moeilijk probleem, net als de veralgemening tot het radiaal vervormd geval en het Cliffordvervormde geval. In dit proefschrift ontwikkelen we de Laplacemethode verder om de expliciete uitdrukking van de Dunklkern in het dihedrale geval te geven. De berekening in het Cliffordgeval verloopt vergelijkbaar.

Nu geven we een overzicht van de inhoud van dit proefschrift.

In hoofdstuk 2 geven we de preliminaire resultaten voor dit proefschrif: dit zijn de Clifford algebra, de fundamentele Euclidische Cliffordanalyse, de Laplacetransformatie (die het belangrijkste instrument is), en de operatorexponentiële in de klassieke Fourieranalyse.

In hoofdstuk 3 introduceren we de Laplacetransformatiemethode om de Clifford-Fourierkern te berekenen. Dit wordt gedaan door de invoering van een hulpveranderlijke t waarna de klassieke Fourierkern door de Cauchykern en de Szegökern in het Laplacedomein uitgedrukt wordt. De inwerking van de exponentiële operator wordt gerealiseerd via de monogene uitbreiding van de twee kernen. We verkrijgen door Laplace inversie een gesloten uitdrukking in het evendimensionale geval. Bovendien zijn we in staat om de genererende functie van de evendimensionale kernen berekenen. Deze nieuwe methode reproduceert ook de vlakkegolfexpansies van de kernen die verkregen werden via de vroegere methodes. Voor de oneven dimensies worden nu integraalformules opgesteld.

In hoofdstuk 4 ontwikkelen we de Laplacetransformatiemethode verder om de verkregen Fourierkern in [28] te berekenen via de representatietheorie van $\mathfrak{sl}_2$. Eerst stellen we het verband op tussen de kern van de Clifford-Fourier transformatie en de algemene Clifford-Fouriertransformatie. Via die relatie geven we de expliciete uitdrukking van de kern en de genererende functie van de evendimensionale kernen. Verder bepalen we veeltermen G die tot polynomiaal begrensde kernen aanleiding geven. Dit is interessant omdat het nieuwe perspectieven opent voor het definiëren van onevendimensionale hypercomplexe Fouriertransformaties met behoorlijk begrensde kernfuncties.

In hoofdstuk 5 concentreren we ons op de ontwikkeling van de Clifford-Fouriertransformatie in de hyperbolische ruimte. De voorstelling en de analyse van signalen in niet-Euclidische meetkunde is een terugkerend probleem in veel wetenschappelijke gebieden. Veel pogingen werden al gewijd aan dit vraagstuk, zie [1, 2, 10]. De nieuwe transformatie wordt gedefinieerd door de exponentiële inwerking van de sferische Dirac operator op de Helgason Fourierkern, die gebaseerd is op meetkundige eigenschappen, d.w.z. de geodetische bol is er een Euclidische bol. Uitgaande van het verband tussen de Laplacedomeinresultaten van het Euclidische en het hyperbolische geval verkrijgen we expliciete uitdrukkingen voor alle evendimensionale kernen.

In hoofdstuk 6 ontwikkelen we de Laplacemethode om expliciete en integraaluitdrukkingen te verkrijgen voor de kern van de (κ, a)-veralgemeende Fouriertransformatie voor $\kappa = 0$. Voor de dihedrale groep wordt deze methode ook toegepast op de Dunklkern evenals de Dunkl Besselfunctie. Door gebruik van de Poissonkern neemt de kern in het Laplacedomein een veel eenvoudiger vorm aan. De inverse Laplacetransformatie wordt dan berekend met behulp van de veralgemeende Mittag-Lefflerfunctie om integraaluitdrukkingen te verkrijgen. Indien de betrokken parameters gehele getallen zijn, worden expliciete formules verkregen via splitsing in partieelbreuken. Nieuwe bovengrenzen voor de kern van de (κ, a) - veralgemeende Fouriertransformatie worden eveneens verkregen.

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List of Symbols

$[\cdot,\cdot]$	Minkowski inner product
$\bar{\ }$	conjugation
$\binom{\cdot}{\cdot}$	binomial coefficient
$\cos$	cosine
Δ	Laplace operator
Δ_κ	Dunkl Laplace operator
Δ_{H^m}	Laplace-Beltrami operator on H^m
$\Gamma(x)$	gamma function
Γ_x	angular Dirac operator in x
$\langle\cdot,\cdot\rangle$	Euclidean inner product
$\mathbb{N}$	natural number
$\mathbb{R}^{1,m}$	Minkowski space
$\mathbb{R}^m$	Euclidean space of dimension m
$\mathbb{S}$	spinor space
$\mathcal{C}\ell_{0,m}$	real Clifford algebra
$\mathcal{C}\ell_{0,m}^{(l)}$	real l-vector
$\mathcal{F}$	Fourier transform
$\mathcal{H}_j(\upsilon_\kappa)$	j-Dunkl harmonic polynomial
$\mathcal{H}_k$	k-harmonic polynomial

$\mathcal{L}$	Laplace transform
$\mathcal{L}^{-1}$	inverse Laplace transform
$\mathcal{M}_k$	k-monogenic polynomial
$\mathcal{P}_k$	k-homogeneous polynomial
$\mathcal{R}$	root system
$\mathcal{S}(\mathbb{R}^m)$	Schwartz space
Im	Imaginary part
Re	real part
∇	gradient operator
ω_m	surface area of the unit ball
$\phi_{j,k,l}$	Hermite function
sin	Sine
$B(k_1, k_2)$	beta function
B^m	unit ball in $\mathbb{R}^m$
$c(\lambda)$	Harish-Chandra function
$C_k^{(\lambda)}(x)$	Gegenbauer polynomial
D	Dirac operator
dx	volume element
$E_{\epsilon,\gamma}^{\delta}(z)$	generalized Mittag-Leffler function
$E_\kappa(x, y)$	Dunkl kernel
e_j	basis element of m-dimensional real space
E_x	Euler operator
H^m	Hyperbolic space of dimension m
I	identity matrix

J_k	Bessel function of first kind
$O(m)$	orthogonal group
S^{m-1}	$m-1$-sphere
$SO(1,m)$	Lorentz group
$SO(m)$	special orthogonal group
T_j	Dunkl operator

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